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NONLINEAR GALERKIN METHOD AND CRANK-NICOLSON METHOD FOR VISCOUS INCOMPRESSIBLE FLOW*

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Abstract

In this article we discuss a new full discrete scheme for the numerical solution of the Navier-Stokes equations modeling viscous incompressible flow. This scheme consists of nonlinear Galerkin method using mixed finite elements and Crank-Nicolson method. Next, we provide the second-order convergence accuracy of numerical solution corresponding to this scheme. Compared with the usual Galerkin scheme, this scheme can save a large amount of computational time under the same convergence accuracy.

Key words: Nonlinear Galerkin method, Crank-Nicolson method, Viscous incompressible flow.

1. Introduction

Nonlinear Galerkin method is numerical method for dissipative evolution partial differential equations where the spatial discretization relies on a nonlinear manifold instead of a linear space as in the classical Galerkin method. More precisely, one considers a finite dimensional space $V_h - h$ being some parameter related to the spatial discretization$-$which is splitted as $V_h = V_H + W_h$, where $H \gg h$ and W_h is a convenient supplementary of V_H in V_h. One looks for an approximate solution u^h lying in a maniflod $\Sigma = \text{graph}\phi$ of $V_h; u^h$ takes the form $u^h = v^H + \phi(v^H)$ where v^H lies in V_H and ϕ is a mapping from V_H into W_h. The method reduces to an evolution equation for v^H, obtained by projecting the equations under consideration on the manifold $\Sigma = \text{graph}\phi$. The related works see [1, 2, 3]. In a classical Galerkin method, typically, we have $\phi = 0$.

The papers[2,3] have extended the nonlinear Galekrin method to the Navier-Stokes equations in the framework of mixed finite elements. However, the paper[2] does not deal with the case of time discretization and the paper[3] only obtains the first-order convergence accaracy for time discretization. Our purpose here is to modify the approximate scheme of [2] and consider the discretization with respect to time of the modified scheme by the Crank-Nicolson method[4]. Also, we aim to derive the full second-order convergence accuracy of numerical solution corresponding to this full discrete scheme. Finally, we compare the full discerete scheme with the usual Galerkin scheme, which shows that the new full discrete scheme is more simple than the usual Galerkin scheme.

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2. The Navier-Stokes Equations

Let Ω be a bounded domain in R^2 assumed to have a Lipschitz-continuous boundary Γ. We consider the time-dependent Navier-Stokes equations describing the flow of a viscous incompressible fluid confined in Ω:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad &\text{in} \quad \Omega\times R^+ \\ \operatorname{div} u = 0 \qquad &\text{in} \ \ \Omega\times R^+ \\ u = 0 \qquad &\text{on} \ \ \Gamma\times R^+ \\ u(0) = u_0 \qquad &\text{in} \ \ \Omega \end{aligned} \tag{2.1}$$

where $u = (u_1, u_2)$ is the velocity, p is the pressure, f represents the density of body force, $\nu > 0$ is the viscosity and u_0 is the initial velocity with $\operatorname{div} u_0 = 0$.

In order to introduce a variational formulation, we set

$$Y = L^2(\Omega)^2, M = L_0^2(\Omega) = \Big\{q \in L^2(\Omega); \int_\Omega q dx = 0\Big\}$$

We denote by $(\cdot,\cdot), |\cdot|$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^2$ and identify $L^2(\Omega)$ with its dual space. We set

$$Au = -\nu\Delta u, \quad B(u,v) = (u\cdot\nabla)v + \frac{1}{2}(\operatorname{div} u)v$$

It is well known that A is a linear unbounded self-adjoint operator in Y with domain $D(A) = (H^2(\Omega)\cap H_0^1(\Omega))^2$ dense in Y; and A is positive closed and the inverse A^{-1} of A is compact, self-adjoint in Y. We then can define the powers A^s of A for any $s \in R$; the space $D(A^s)$ is a Hilbert space when endowed with the scalar product $(A^s\cdot, A^s\cdot)$ and norm $|A^s\cdot|$. We set

$$X = D(A^{\frac{1}{2}}) = H_0^1(\Omega)^2, \|\cdot\| = |A^{\frac{1}{2}}\cdot|, ((\cdot,\cdot)) = (A^{\frac{1}{2}}\cdot, A^{\frac{1}{2}}\cdot)$$

Next, we define the bilinear forms

$$\begin{aligned} a(u,v) &= \nu\langle Au, v\rangle \quad \forall u, v \in X \\ D(v,q) &= (q, \operatorname{div} v) \quad \forall v \in X, q \in M \end{aligned}$$

and the trilinear form

$$b(u,v,w) = \langle B(u,v), w\rangle \quad \forall u,v,w \in W$$

So, we obtain the variational formulation of problem (2.1):

For any $t > 0$, find a pair $(u(t), p(t)) \in X\times M$ such that

$$\begin{aligned} &(u_t, v) + a(u,v) + b(u,u,v) - D(v,p) = (f,v) \quad \forall v \in X \\ &D(u,q) = 0 \quad \forall q \in M \end{aligned} \tag{2.2}$$