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THE FINITE ELEMENT METHOD FOR SEMILINEAR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS*[1)]

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Abstract

In this paper we investigate the existence, uniqueness and regularity of the solution of semilinear parabolic equations with coefficients that are discontinuous across the interface, some prior estimates are obtained. A net shape of the finite elements around the singular points was designed in [7] to solve the linear elliptic problems , by means of that net, we prove that the approximate solution has the same convergence rate as that without singularity.

Key words: Finite element, Semitinear parabolic equation, Discontinuous coefficients.

1. Introduction

Let x be points on plane R^2, and Ω be a polygonal domain, we denote the boundary of Ω by $\partial\Omega$. There in Ω are finite many broke lines which divide it into finite polygonal subdomains $\Omega_l, l = 1, \cdots, L$. The function $p(x) \in L^2(\Omega)$ is assumed to have bounded first derivatives in all subdomains Ω_l, while p is allowed to be discontinuous on the interfaces $\partial\Omega_i \cap \partial\Omega_j$. And there exists a positive constant τ such that

$$p(x) \geq \tau, \quad \forall x \in \Omega.$$

We adopt the usual notations of the Sobolev spaces in this paper, that is, denote by $H^s(\Omega)$ and $H_0^s(\Omega)$ the spaces and $\|\cdot\|_s$ the norms, $|\cdot|_s$ the semi norms.

We define a linear operator A by

$$Au = \bigtriangledown(p(x) \bigtriangledown u), \quad D(A) = \{u \in H_0^1(\Omega), Au \in L^2(\Omega)\},$$

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where $\bigtriangledown$ is gradient operator. Let $f(x) \in L^2(\Omega)$, $u_0 \in D(A^2), T_0$ be a positive constant. In this paper we study the following nonlinear initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au - \phi(u) + f(x) & \text{in} \quad \Omega \times (0, T_0] \\ u(t,x) = 0 & texton \quad \partial\Omega \times [0, T_0] \\ u(0,x) = u_0(x) & \text{in} \quad \Omega \end{cases} \tag{1.1}$$

where $\phi \in C^1(R)$ and we assume further that there exist positive constants λ_1, λ_2 and positive integer k such that

$$0 < \phi'(u) \leq \lambda_1 \mid u \mid^k + \lambda_2.$$

In section 2 we first get the existence and uniqueness of (1.1), then we investigate the regularity of the solution, some prior estimates are obtained. In section 3 we present the finite element method which is suitable for (1.1), some error estimates for interpolation operator of finite element space are gotten. In section 4 we obtain the error estimate for the finite element solution.

2. Existence, Uniqueness and Regularity

By the usual approach[1−3] it is easy to prove that A is the infinitesemal generator of an analytic semigroup $T(t)$ on $L^2(\Omega)$. As in [1], [2] and [3], for $0 \leq \alpha \leq 1$, we introduce operators A^α which are fractional powers of A, we denote the domain of A^α by $D(A^\alpha)$, $D(A^\alpha)$ equipped with the norm $\|u\|_\alpha = \|A^\alpha u\|_{L^2(\Omega)}$ is a Banach space which we denote by X_α.

By Gagliardo-Nirenberg inequality[1−3], we have

$$X_\alpha \subset L^{4k}(\Omega) \quad \text{when} \quad 1 - \frac{1}{2k} < \alpha \leq 1. \tag{2.1}$$

and the imbeddings are continuous.

Analogous to [1], [2] and [3], by the contraction mapping theorem, it is easy to know that (1.1) has a unique local solution $u \in C([0, t_1]; X_\alpha)$, where $1 - \dfrac{1}{2k} < \alpha < 1, t_1$ is a positive constant depending on u_0.

Similar to [4], we can prove that there is a unique $u^* \in D(A)$ satisfying

$$Au^* - \phi(u^*) + f(x) = 0.$$

Now we consider the following initial problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = Av - (\phi(v + u^*) - \phi(u^*)) & \text{in} \quad \Omega \times (0, T_0] \\ v(t,x) = 0 & \text{on} \quad \partial\Omega \times [0, T_0] \\ v(0,x) = u_0(x) - u^*(x) & \text{in} \quad \Omega \end{cases} \tag{2.2}$$

Since as long as the solution exists, $t \to \int_\Omega \mid v(t,x) \mid^{2l} dx$ is nonincreasing for all positive integer l, it follows from (2.1) and Gronwall's inequality that there exists a constant M_1 independent of t such that

$$\|v(t,x)\|_\alpha \leq M_1 \tag{2.3}$$