Journal of Computational Mathematics, Vol.17, No.2, 1999, 199–208.

ON THE LINEAR CONVERGENCE OF PC-METHOD FOR A CLASS OF LINEAR VARIATIONAL INEQUALITIES*

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Abstract

This paper studies the linear convergence properties of a class of the projection and contraction methods for the affine variational inequalities, and proposes a necessary and sufficient condition under which PC-Method has a globally linear convergence rate.

Key words: Affine variational inequality, Projection and contraction method, Linear convergence.

1. Introduction

Let M be an $n \times n$ matrix and let q be a vector in R^n, the n-dimensional Enclidean space. Let Ω be an nonempty closed convex set. The linear variational inequality problem (denoted by (LVI)) is to find $x^* \in \Omega$ such that

$$(x - x^*)^T (Mx^* + q) \geq 0, \quad \forall x \in \Omega. \tag{1.1}$$

The problem (1.1) is well known in optimization and contains as special cases linear (and quadratic) programming, bimatrix game, etc. (see Cottle and Dantzig [1]). When Ω is a polyhedral set, for convenience expressed as

$$X = \{x \in R^n | Ax \geq b\}, A \in R^{m \times n}, b \in R^m, \tag{1.2}$$

it is called the affine variational inequality problem (AVI). When $\Omega = R^n_+$, the nonnegative orthant in R^n, it is again called the linear complementarity problem (LCP). For these subjects, many computational methods and theoretical results have been developed (See Harker and Pang [2], Cottle, Pang and Stone [3], Isac [4] etc.). An important class of methods is the projection-type method, originally proposed by Goldstein [5], Levitin and Polyak [6] for solving convex programming. More recently, He [7–12] has proposed a special class of the projection methods for problem (1.1). The iterative form is as follows. Given $x^k \in R^n$ (or Ω), find the search direction $d(x^k)$ such that it satisfies

$$x^{k+1} = x^k - \alpha_k \cdot d(x^k), \quad \text{or} \quad x^{k+1} = P_\Omega[x^k - \alpha_k d(x^k)], \tag{1.3a}$$

* Received September 3, 1996.

$$\|x^{k+1} - x^*\|_G^2 \le \|x^k - x^*\|_G^2 - \rho_k \cdot \|e(x^k)\|^2, \tag{1.3b}$$

where $\alpha_k > 0$ is the search step length, ρ_k is a positive number, $G \in R^{n\times n}$ is symmetric and positive definite, $\|x\|_G = (x^T G x)^{\frac{1}{2}}, P_\Omega[\cdot]$ denotes the projection from R^n onto Ω, i.e.,

$$P_\Omega[x] = \arg\min\{\|x - y\| \quad |\forall y \in \Omega\}, \tag{1.4}$$

$e(x) = x - P_\Omega[x - (Mx + q)]$, and $x^* \in \Omega^*$, which denotes the set of solutions of problem (1.1). From (1.3) we readily see that the sequence $\{\|x^k - x^*\|_G^2\}$ has a contractive property. Therefore, He defines this class of methods as the projection and contraction method (PC-Method). The main advantages of the method are its simplicity, robustness and ability to handle the large-scale problems.

In [12], He has summerized the basic idea of finding the search direction $d(x)$ of PC-Method, i.e., for any $x^* \in \Omega^*$, it holds that

$$(x - x^*)^T d(x) \ge r \cdot \|e(x)\|^2, \quad r > 0, \tag{1.5}$$

and proven that the PC methods of He [7–11] are all globally convergent for varieties of monotone problems. However, He only prove that PC-Method is globally linearly convergent for the monotone linear complementarity problem.

The purpose of this paper is to develop the linear convergence theory of PC-Method. The main results obtained in this paper are as follows.

(a) For the monotone problem (AVI), a class of PC methods is linearly convergent. Furthermore, $x^k \to x^*$ Q-linearly, $\|e(x^k)\| \to 0$ R-linearly.

(b) For strongly monotone problem (AVI), the necessary and sufficient condition under which a class of PC methods has linearly convergent rate is the search direction $d(x)$ to be strongly descent (see Theorem 4.2).

This paper is organized as follows. In section 2, we give the definitions of the strictly descent direction and strongly descent direction, and discuss their convergence properties, which extend the previous convergence theory. In Section 3, we investigate the linear convergence of PC-Method when it is applied to solve the monotone problem (AVI). Finally, Section 4 considers the special case of (AVI) where M is positive definite.

We adopt the following notations throughout. For any $x \in R^n$ and $y \in R^n$, we denote by $x^T y$ the Euclidean inner product of x with y. For any $x \in R^n$, we define $\|x\| = (x^T x)^{\frac{1}{2}}$. For any $C_1, C_2 \subseteq R^n$, we denote by $\text{dist}(C_1, C_2)$ the usual Euclidean distance between two sets C_1 and C_2, that is,

$$\text{dist}(C_1, C_2) = \inf\{\|x - y\| \,|x \in C_1, y \in C_2\}.$$

For any symmetric matrix $A \in R^{n\times n}$, we denote by $\lambda_{\min}(A)$ (and $\lambda_{\max}(A)$) the minimum (and maximum) eigenvalue of A. Other notations have the usual meaning. Throughout this paper we assume that (H_1) $\Omega^* \ne \phi$, and (H_2) M is positive semidefinite (but not necessarily symmetric).