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FINITE ELEMENT ANALYSIS OF A LOCAL EXPONENTIALLY FITTED SCHEME FOR TIME-DEPENDENT CONVECTION-DIFFUSION PROBLEMS[*1)]

Xing-ye Yue
(*Department of Mathematics, Suzhou University, Suzhou 215006, China*)
Li-shang Jiang
(*Institute of Mathematics, Tongji University, Shanghai 200092, China*)
Tsi-min Shih
(*Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong*)

Abstract

In [16], Stynes and O'Riordan(91) introduced a local exponentially fitted finite element (FE) scheme for a singularly perturbed two-point boundary value problem without turning-point. An ε-uniform $h^{1/2}$-order accuracy was obtain for the ε-weighted energy norm. And this uniform order is known as an optimal one for global exponentially fitted FE schemes (see [6, 7, 12]).

In present paper, this scheme is used to a parabolic singularly perturbed problem. After some subtle analysis, a uniformly in ε convergent order $h|\ln h|^{1/2}+\tau$ is achieved (h is the space step and τ is the time step), which sharpens the results in present literature. Furthermore, it implies that the accuracy order in [16] is actuallay $h|\ln h|^{1/2}$ rather than $h^{1/2}$.

Key words: Singularly perturbed, Exponentially fitted, Uniformly in ε convergent, Petrov-Galerkin finite element method.

1. Introduction

Consider the time-dependent convection-diffusion problem

$$u_t-\varepsilon u_{xx}+a(x,t)u_x+b(x,t)u=f(x,t),\quad (x,t)\in[0,1]\times[0,T] \tag{1.1}$$

$$u(0,t)=u(1,t)=0,\quad t\in[0,T], \tag{1.2}$$

$$u(x,0)=u_0(x),\quad x\in[0,1], \tag{1.3}$$

$$a(x,t)\geq\alpha>0, \tag{1.4}$$

$$b(x,t)-a_x(x,t)/2\geq\beta>0, \tag{1.5}$$

where $0\leq\varepsilon\ll 1$. (1.1)-(1.5) can be regarded as a parabolic singularly perturbed problem. In general, the solution has a boundary layer at the outflow boundary $x=1$. See [1] and [15] for discusss of the properties of $u(x,t)$.

Such problems are all pervasive in applications of mathematics to problems in the science and engineering. Among these are the Navier-Stokes equation of fluid flow

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at high Reynolds number, the drift-diffusion of semiconductor, the mass conservation law in porous mediam. They have mainly hyperbolic nature for ε is small. This makes them difficult to solve numerically. It's well know that classical methods do not work well for (1.1)–(1.5) (see [3, 10]). The main problem is how to construct an ε-uniformly convergent scheme. Many authors have suggested various methods to solve such problems, see [2, 5, 9, 10, 13] and their references for the discussion of finite difference methods.

As to ε-uniformly convergent FE scheme, Gartland [4], Stynes and O'Rriordan [14, 16], Guo [6–8] and Sun & Stynes [17] have constructed quite a few methods. Guo 93 [8] proved that any scheme on a uniform mesh for (1.1)–(1.5) that was globally L^∞ convergent uniformly in ε, could not only have polynomial coefficients; the coefficients must depend on exponentials. But for highly nonequidistant meshes, such as Shiskin-type meshes, standard polynomial FE methods can also yield ε-uniformly convergent results (see Th 2.54 of [12]).

In the following, we'll focus on a scheme suggested by Stynes and O'Riordan 91 [16] for a steady-case of (1.1)–(1.5), which we call as "local exponentially fitted FE scheme". They used exponentially fitted splines in the boundary layer region and outside it, the normal continuous piecewise linear polynomials instead. An ε-uniform convengence order $h^{1/2}$ was obtained. Although this order is known as an optimal one for global exponentially fitted FE schemes, we can sharpen it to order $h|\ln h|^{1/2}$ in the case of local exponential fitting as a corollary of our main result for (1.1)–(1.5).

2. The Local Exponentially Fitted FE Scheme

Before describing the scheme, we need to know the behavior of the solution u of (1.1)–(1.5). Just for simplicity, we assume that $a(x,t), b(x,t), f(x,t)$ and $u_0(x)$ are sufficiently smooth and satisfy necessary compatibility assumptions on the corners of the boundary. Then we have the following lemma.

Lemma 2.1[15]**.** (1.1)–(1.5) *has a unique smooth solution $u(x,t)$ which satisfies*

$$|\partial_x^i \partial_t^j u(x,t)| \le C[1+\varepsilon^{-i} e^{-\alpha(1-x)/\varepsilon}] \quad \forall (x,t) \in [0,1]\times[0,T], \tag{2.1}$$

for $0 \le i \le 1$ and $0 \le i+j \le 2$.

Throughout this paper, C will denote a generic positive constant independent of ε.

We work with an arbitrary tensor product grid on $[0,1]\times[0,T]$. In the x-direction, let $0 = x_0 < x_1 < \cdots < x_N = 1$, with $h_i = x_i - x_{i-1}$ for $i = 1, \cdots, N$, and set $h = \max_i h_i$, $\bar{h}_i = (h_i + h_{i+1})/2$.

We assume that

$$\frac{h}{h_i} \le C \quad \forall i = 1, \cdots, N.$$

In the t-direction, let $0 = t_0 < t_1 < \cdots < t_M = T$, with $\tau_m = t_m - t_{m-1}$, for $m = 1, 2, \cdots, M$ and $\tau = \max_m \tau_m$.

Assuming $2\varepsilon|\ln\varepsilon|/\alpha < 1/2$ (it is not a restriction for ε is small), and set

$$K = \max\{i : 1 - x_i \ge 2\varepsilon|\ln\varepsilon|/\alpha\}. \tag{2.2}$$

From lemma 2.1, we have