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A NONLINEAR GALERKIN METHOD WITH VARIABLE MODES FOR KURAMOTO-SIVASHINSKY EQUATION[*1)]

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Abstract

This article proposes a kind of nonlinear Galerkin methods with variable modes for the long-term integration of Kuramoto-Sivashinsky equation. It consists of finding an appropriate or best number of modes in the correction of the method. Convergence results and error estimates are derived for this method. Numerical examples show also the efficiency and advantage of our method over the usual nonlinear Galerkin method and the classical Galerkin method.

Key words: Kuramoto-Sivaskinsky equation, Nonlinear Galerkin method, Approximation, Convergence

1. Introduction

The nonlinear Galerkin method was introduced by Marion and Temam[4], which is stemmed from the theory of inertial manifolds and dynamical system theory. The considerable increase in the computing power during last years makes it possible for the mathematicians to solve numerical problems for approximating various dissipative evolution equations on large interval of time. Indeed, the nonlinear Galerkin method has proven to be a powerful tool for such problems (See [9], [11] and references therein).

Recently, this method has been applied to the long time integration of Kuramoto-Sivashinsky equation[12]. Thanks to a newly established inequality for the nonlinear term of Kuramoto-Sivashinsky equation, we can extend the method to a nonlinear Galerkin method with variable modes. Here the method involves a changeable number for the small-scale components $z_s = z_{s(m)}$, when the unknown function is $u \approx u_m + z_s$. After the analysis of error estimates we give an optimal value of s or $\omega = m + s$ which reduces the order of the error of the method to the lowest.

This paper is organized as follows: Section 2 contains the description of the equation and some preliminary results. In Section 3 we describe the modification of nonlinear Galerkin method with variable modes and prove successively the convergence of the method. In Section 4 we state and prove the error estimates of the method and give the possible minimum modes for the method. Finally, in Section 5 we make comparisons of various numerical computations for two examples which show a significant gain in computing time for our method.

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2. The Equation and Its Functional Setting

The Kuramoto-Sivashinsky equation with an initial condition and a periodic boundary condition reads as follows (with dimension= 1 and period= l):

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^2 u}{\partial x^2} + u\dfrac{\partial u}{\partial x} = 0 & 0 < x < l, \quad t > 0 & (2.1) \\ u(x,0) = u_0(x) & 0 \le x \le l & (2.2) \\ u(x,t) = u(x+l,t) & t \ge 0 & (2.3) \end{cases}$$

For the functional setting of the equation, we can rewrite this partial differential equation into an abstract evolution equation in a Hilbert space H with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. In this case, we have $H = \{u | u \in L^2(0,l),\ u(0,t) = u(l,t) = 0\}$. Thus the equations (2.1)–(2.3) become

$$\begin{cases} \dfrac{du}{dt} + Au + B(u) + Cu = f & (2.4) \\ u(0) = u_0 & (2.5) \end{cases}$$

Here, we set $A = \dfrac{\partial^4}{\partial x^4}$, $B(u) = u\dfrac{\partial u}{\partial x}$ and

$$Cu = \begin{cases} \dfrac{\partial^2 u}{\partial x^2} & l < 2\pi \\ \dfrac{\partial^2 u}{\partial x^2} + \phi\dfrac{\partial u}{\partial x} + \phi' u & l \ge 2\pi \end{cases}$$

$$f = \begin{cases} 0 & l < 2\pi \\ -\phi^{(4)} - \phi'' - \phi\phi' & l \ge 2\pi \end{cases}$$

where $\phi = \phi(x)$ is a function given in [5] to keep the coercivity property of the operator $A + C$.

Since A^{-1} is compact and self-adjoint, there exists an orthonormal basis of H which consists of the eigenvectors of A: $Aw_j = \lambda_j w_j$, $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lambda_j \to \infty$ as $j \to \infty$.

Given another Hilbert space V endowed with scalar product $((\cdot,\cdot))$ and norm $\|\cdot\|$, $V = H_p^2(0,l) \cap H$. We denote the domain of the operator A by $D(A) = H_p^4(0,l) \cap H$. And we know that $B(u) = B(u,u)$ is a bilinear operator from $V \times V$ into V', C is a linear operator from V into H and $f \in H$.

Define a trilinear form b on V by $b(u,v,w) = \langle B(u,v), w\rangle_{V',V} \quad \forall u,v,w \in V$, we recall the following well-known properties:

$$b(u,u,u) = 0 \quad \forall u \in V \tag{2.6}$$

$$|b(u,v,w)| \le c_1 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2} \quad \forall u,v,w \in V \tag{2.7}$$

$$|Cu| \le c_2 \|u\| \quad \forall u \in V \tag{2.8}$$

$$|B(u,v)| \le c_3 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} \quad \forall u \in V, v \in D(A) \tag{2.9}$$

$$|B(u,v)| \le c_4 |u|^{1/2} |Au|^{1/2} \|v\| \quad \forall u,v \in D(A) \tag{2.10}$$

$$|B(u,v)| \le c_5 \Big(1 + \log\frac{|Au|^2}{\lambda_1 \|u\|^2}\Big)^{1/2} \|u\| \|v\| \quad \forall u \in D(A), v \in V \tag{2.11}$$