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A KIND OF IMPLICIT ITERATIVE METHODS FOR ILL-POSED OPERATOR EQUATIONS*1)

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Abstract

In this paper we propose a kind of implicit iterative methods for solving ill-posed operator equations and discuss the properties of the methods in the case that the control parameter is fixed. The theoretical results show that the new methods have certain important features and can overcome some disadvantages of Tikhonov-type regularization and explicit iterative methods. Numerical examples are also given in the paper, which coincide well with theoretical results.

Key words: Ill-posed equation, Implicit iterative method, Control parameter, Discrepamcy principle, Optimal convergence rate

1. Introduction

Let X, Y be two real Hilbert spaces and let $A : X \to Y$ be a bounded linear operator. Consider the operator equation

$$Ax = y. \tag{1.1}$$

If $R(A)$, i.e., the range of A, is nonclosed in Y, equation (1.1) is ill-posed[1]. Many important problems in applied sciences result in this kind of equations[2,3]. In this paper we consider the Moore-Penrose generalized solution $x^+ = A^+y$ to equation(1.1), where A^+ is the Moore-Penrose generalized inverse of operator A[1]. A^+y exists if and only if y $\in D(A^+) = R(A) + R(A)^\perp$. In practice, instead of (1.1) we usually only have a perturbed version of equation

$$Ax = y_\delta, \tag{1.2}$$

where the perturbed right-hand term $y_\delta \in B_\delta(y) = \{z \in Y | \|Q(z-y)\| \le \delta\}$ with $\delta > 0$ being a known error level and Q being the orthogonal projective operator from Y onto $\overline{R(A)}$. A well-known kind of methods to obtain a suitable approximation of x^+ by using the perturbed equation (1.2) are regularization methods which can be constructed by variation methods or the spectrum of operator A^*A[1,2]. Another usually used approach is iterative method. In 1951, Landweber[4] proposed the first iterative method to solve ill-posed operator equations, though the convergence rate of the method is very slow. The next breakthrough was made by Nemirovskii and Palyak[5] and Brakhage[6] who developed independently iterative procedures of so-called ν-method. In [7], Hanke

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analysed all above-mentioned methods and some others, and established a framework for explicit iterative methods. However, the explicit iterations discussed in [7] still have some disadvantages.

We will discuss the following kind of implict iterative methods for equation (1.1)

$$(A^*A+\alpha_k I)x_k = A^*y+\alpha_k x_{k-1}, \quad k=1,2,\cdots \qquad (1.3)$$
$$x_0 \text{ given,}$$

where $\alpha_k > 0$ are some parameters and $A^*: Y \to X$ is the adjoint operator of A. In this paper, we assume all α_k are equal and hence (1.3) becomes

$$(A^*A+\alpha I)x_k = A^*y+\alpha x_{k-1}, \quad k=1,2,\cdots \qquad (1.4)$$

In a relative paper we will consider the general case.

2. Convergence Properties for Nonperturbed Equation(1.1)

Iteration (1.4) may be rewritten as

$$(A^*A+\alpha I)(x_k - x_{k-1}) = A^*(y-Ax_{k-1}) \qquad (2.1)$$

Let $r_k = A^*(y-Ax_k)$, and (2.1) becomes $x_k = x_{k-1}+(A^*A+\alpha I)^{-1}r_{k-1}$. Repeat use of this formula gives $r_k = \alpha^k(A^*A+\alpha I)^{-k}r_0$ and

$$x_k = U_{k,\alpha}(A^*A)A^*y + P_{k,\alpha}(A^*A)x_0 \qquad (2.2)$$

with

$$P_{k,\alpha}(\lambda) = \Big(\frac{\alpha}{\lambda+\alpha}\Big)^k \qquad (2.3)$$
$$U_{k,\alpha}(\lambda) = (1-P_{k,\alpha}(\lambda))/\lambda \qquad (2.4)$$

In the paper we will use following notations:

I_o $:= (0, \|A^*A\|]$

S $:= \{x \in X | A^*Ax = A^*y\}$

P_s : the orthogonal projection $X \to S$

$\{E_\lambda\}$ and $\{F_\lambda\}$: the spectrum families of self-adjoint operators A^*A and AA^*, respectively.

S is the set of the least squares solutions to equation (1.1) and $S \neq \emptyset$ if $y \in D(A^+)$. In the sequel we will always assume the case and Qy≠0 as well.

Lemma 2.1. *For any fixed $\alpha > 0$ and $x \in N(A)^\perp$,*

$$\|P_{k,\alpha}(A^*A)\| \le 1, \ P_{k,\alpha}(A^*A)x \to 0, \ \text{as } k \to \infty.$$

Theorem 2.2. *Let $\{x_k\}$ be the iterates of* (2.1)*, then $x_k \to P_s x_0$, as $k \to \infty$. Especially if $x_0 = 0, x_k \to A^+y$.*

Proof. Since $P_s x_0 \in S$, $A^*y = A^*AP_s x_0$ and $x_0 - P_s x_0 \in N(A)^\perp$. By (2.2) and Lemma 2.1,

$$x_k - P_s x_0 = P_{k,\alpha}(A^*A)(x_0 - P_s x_0) \to 0, \quad \text{as} \quad k \to \infty$$