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STABILITY ANALYSIS OF FINITE ELEMENT METHODS FOR THE ACOUSTIC WAVE EQUATION WITH ABSORBING BOUNDARY CONDITIONS (PART I)*1)

Xiu-min Shao Zhi-ling Lan
(*Institute of Mathematics, Academy of Sciences, Beijing 100080, China*)

Abstract

In Part I and Part II of this paper initial-boundary value problems of the acoustic wave equation with absorbing boundary conditions are considered. Their finite element-finite difference computational schemes are proposed. The stability of the schemes is discussed and the corresponding stability conditions are given. Part I and Part II concern the first- and the second-order absorbing boundary conditions, respectively. Finally, numerical results are presented in Part II to show the correctness of theoretical analysis.

Key words: Stability, Finite element methods, Wave equation, Absorbing boundary conditions

1. Introduction

In the numerical simulation of wave propagation in unbounded or semi-unbounded medium it is necessary to introduce artificial boundaries to obtain finite computational regions. Then some boundary conditions have to be imposed on these boundaries, which should eliminate the reflection of waves at artificial boundaries, so that the obtained solutions rather accurately simulate the solutions in the unbounded domains. (That is why they are called absorbing boundary conditions). The conditions on the artificial boundaries should also guarantee the well-posedness of solutions to the differential equations, which is a necessary condition for the stability of the finite difference or the finite element approximations.

In recent thirty years, a variety of absorbing boundary conditions for wave equations have been developed (see [1]). What is most widely used was given by Clayton and Engquist[2], Engquist and Majda [3,4], based on the pseudodifferential operator theory. A hierarchy of differential boundary conditions was derived to approximate the boundary conditions of the pseudodifferential operator forms. Let the artificial boundary be $x = 0$, and the domain be $t \geq 0, x \leq 0$. For the acoustic wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.1}$$

the mentioned conditions are the followings:

$$\mathcal{B}_1 u|_{x=0} = \Big(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\Big)\Big|_{x=0} = 0,$$

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$$\mathcal{B}_2 u|_{x=0} = \Big(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t \partial x} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\Big)\Big|_{x=0} = 0, \tag{1.2}$$

$$\mathcal{B}_{N+1} u|_{x=0} = \Big(\frac{\partial}{\partial t}\mathcal{B}_N u - \frac{1}{4}\frac{\partial^2}{\partial y^2}\mathcal{B}_{N-1} u\Big)\Big|_{x=0} = 0.$$

The corresponding conditions for the elastic wave equations are complicated, and we are not going to write them here.

In [3], the well-posedness of (1.2) (i.e., the Clayton-Engquist-Majda conditions for the acoustic wave equation) when $N \leq 3$ has been proved. In [5], the authors of this paper generalize (1.2) to the anisotropic elastic wave equations and have proved that the Clayton-Engquist-Majda conditions for the elastic wave equations are ill posed when $N \geq 2$.

In this paper, only the acoustic wave equation

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{C^2(x,y)}\frac{\partial^2 u}{\partial t^2} = f(x,y,t) \tag{1.3}$$

is discussed. But some conclusions are significant also for other wave equations.

In numerical computations, the equation (1.3) with absorbing boundary conditions is approximated usually by finite difference schemes, and seldom by finite element approaches. The author of [6] affirmed that the main difficulty comes from the order of the boundary conditions for which it is not easy to derive a weak formulation which provides a suitable energy estimate. In [6], therefore, a third-order energy is introduced, and a first-order hyperbolic system of 7 unknowns is derived, for which finite element methods can be applied. Obviously, this approach is not desirable for practical computation.

In this paper, finite element-finite difference schemes for the equation (1.3) with the first and second order absorbing boundary conditions of (1.2) are proposed. Their stability is discussed, and the stability conditions are given. The Part I is devoted to the first order absorbing boundary condition, and the Part II to the second order boundary condition. The numerical results are presented in the Part II, which show the correctness of the theoretical conclusions.

For the sake of simplicity, we shall restrict ourselves to the two-dimensional case. The three-dimensional case can be discussed similarly without any difficulty.

2. Finite Element-Finite Difference Schemes

Let the computational domain be $\overline{\Omega}, \Omega = \{(x,y) : -a < x < a, 0 < y < b\}$; $\Gamma_1 = \{(x,y)\text{: } -a \leq x \leq a, y = 0\}$ be a natural boundary, and $\partial\Omega' = \partial\Omega/\Gamma_1$ be the artificial boundary.

Introduce the inner product notations

$$(u,v) = \int\int_\Omega uv dx dy, \quad \langle u,v\rangle = \int_{\partial\Omega'} uv ds.$$

Define the space $H^{1,0}(\Omega) = \{v(x,y) \in H^1(\Omega) : v|_{\Gamma_1} = 0\}$. It is obvious that $H^{1,0}(\Omega)$ is a closed subspace of $H^1(\Omega)$.

In the following discussion, let n denote outer normal direction, and s tangential direction of the boundary $\partial\Omega'$. Suppose that in (1.3), $C(x,y) \in L^\infty(\Omega)$ and $C(x,y) > 0$;