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A FINITE DIMENSIONAL METHOD FOR SOLVING NONLINEAR ILL-POSED PROBLEMS*[1)]

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Abstract

We propose a finite dimensional method to compute the solution of nonlinear ill-posed problems approximately and show that under certain conditions, the convergence can be guaranteed. Moreover, we obtain the rate of convergence of our method provided that the true solution satisfies suitable smoothness condition. Finally, we present two examples from the parameter estimation problems of differential equations and illustrate the applicability of our method.

Key words: Nonlinear ill-posed problems, Finite dimensional method, Convergence and convergence rates.

1. Introduction

In this paper we consider the nonlinear problems of the form

$$F(x) = y_0, \tag{1}$$

where $F : D(F) \subset X \to Y$ is a nonlinear operator between real Hilbert spaces X and Y and $y_0 \in R(F)$. The norms in X and Y will be denoted by $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. We are mainly interested in those problems of the form (1) for which the solution does not depend continuously on the right hand side. Such problems are called ill-posed. We refer to [5] for a number of important inverse problems in natural sciences which lead to such ill-posed problems.

Let L be a linear operator

$$L : D(L) \subset X \to Z$$

with Z a Hilbert space (the norm is denoted by $\|\cdot\|_Z$) and $D(F) \cap D(L) \neq \emptyset$. L need not be bounded and allows us to define a seminorm $|\cdot|$ on $D(L)$ by means of $|x| := \|Lx\|_Z$. Let $(x,z)_L := (x,z)_X + (Lx, Lz)_Z$ for each pair $x, z \in D(L)$, then $(\cdot,\cdot)_L$

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is an inner product on $D(L)$, and the induced norm is denoted by $\|z\|_L = \sqrt{(z,z)_L}$ for $z \in D(L)$. If L is closed then $(D(L), \|\cdot\|_L)$ forms a Hilbert space.

Now we choose a concept of "solution" for problem (1). An element $x_0 \in X$ is called an x^*-minimum-seminorm-solution (x^*-MSS) of problem (1) for given $y_0 \in Y$ and $x^* \in D(L)$ if

$$F(x_0) = y_0,$$

and

$$\|Lx_0 - Lx^*\|_Z = \inf\{\|Lx - Lx^*\|_Z \mid F(x) = y_0,\ x \in D(F) \cap D(L)\},$$

where $x^* \in D(L)$ is an *a priori* guess of x_0 and it plays the role of a selection criterion. In the following we always assume the existence of an x^*-MSS x_0 for problem (1). Due to the nonlinearity of F, this solution need not be unique.

Since in practice, we often only know the approximation data y_δ of y_0, $\|y_\delta - y_0\| \le \delta$, regularization technique is required to obtain a reasonable solution to x^*-MSS x_0 due to the ill-posedness of problem (1). Tikhonov regularization is the well known method. In [7], Tikhonov regularization method with the seminorm $|\cdot|$ in the regularization term was introduced and the solution x_α^δ of the minimization problem

$$\min_{x \in D(F) \cap D(L)} \{\|F(x) - y_\delta\|_Y^2 + \alpha\|Lx - Lx^*\|_Z^2\} \tag{2}$$

was used to approximate the x^*-MSS of problem (1). By suitable choice of the regularization parameter α, convergence and convergence rate of x_α^δ were obtained. The practical advantage of allowing for regularization with an operator L is given by the fact that one can realize seminorm regularization terms, which penalize undesired oscillations in the numerical solution without significantly affecting its low modes.

In this paper we will present a finite dimensional method for solving nonlinear ill-posed problems. We describe the method in Section 2 and show that this method is well-defined and prove the existence of the approximate solutions. It is easy to see that our method can be viewed as a modified form of the (generalized) Marti's method if F is a linear operator[10,15]. The analysis for convergence and convergence rates are presented in Section 3 and two examples from the parameter estimation problems of differential equations are given in Section 4 to illustrate the reasonability of our assumptions and the applicability of our method. For the finite-dimensional approximation of Tikhonov regularization of nonlinear ill-posed problems, F is required to be compact[11,13]. For our method, this is not necessary.

2. The Description of the Method

Let x_0 be the sought x^*-MSS of (1). Let $\{P_n\}$ be a sequence of bounded linear operators of finite rank on X such that $P_n x_0 \in D(F) \cap D(L)$ for sufficiently large n, and we can choose positive number sequences $\{b_n\}$ and $\{c_n\}$ such that

$$\|P_n x_0 - x_0\|_X = o(b_n), \quad \lim_{n\to\infty} b_n = 0, \tag{3}$$

$$\|L(P_n x_0 - x_0)\|_Z = O(c_n), \quad \lim_{n\to\infty} c_n = 0. \tag{4}$$