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THE STABILITY AND CONVERGENCE OF COMPUTING LONG-TIME BEHAVIOUR[*1)]

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Abstract

The object of this paper is to establish the relation between stability and convergence of the numerical methods for the evolution equation $u_t - Au - f(u) = g(t)$ on Banach space V, and to prove the long-time error estimates for the approximation solutions. At first, we give the definition of long-time stability, and then prove the fact that stability and compatibility imply the uniform convergence on the infinite time region. Thus, we establish a general frame in order to prove the long-time convergence. This frame includes finite element methods and finite difference methods of the evolution equations, especially the semilinear parabolic and hyperbolic partial differential equations. As applications of these results we prove the estimates obtained by Larsson [5] and Sanz-serna and Stuart [6].

Key words: Stability, Compatibility, Covergence, Reaction-diffusion equation, Long-time error estimates.

1. Introduction

In 1978, Hoff[3] considered the long-time behavior computation of nonlinear reaction-diffusion equations, which is supposed to have an invariant region S, i.e. any local solution arising from a point in S is constrained to lie in. Hoff constructed a family of finit difference schemes for the equations. Under some assumptions he proved that any trajectory starting in S will converge to an asymptotically stable equilibrium, and S is also an invariant region of the difference equations. So Hoff obtained error estimates uniform in time for the difference equations. In 1989, Larsson[5] studied the long-time error estimates of finite-element approximations of reaction-diffusion equation (below dimension 3). The distinction between [5] and [3] is that Larsson didn't assume the equation has a invariant region but has an asymptotically stable hyperbolic equilibrium, and so the trajectories constrict to some neighbourhood of the equilibrium.

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However, by the standard finite time error estimates one can show that the discretization solution will enter this neighbourhood. In 1992, Sanz-Serna and Stuart[6] obtained an error estimates uniform in time for explicit difference scheme of one-dimensional reaction-diffusion equation by using analogous technique of [5]. Note that all of above results are obtained under the condition that the continuous trajectories converge to an asymptotically stable, hyperbolic equilibrium and they are difficult to be generalized.

In this paper, we'll establish the relation between stability and convergence, and then obtain a sufficient condition of long-time convergence of discrete methods for more general equation. Under such a condition we get an error estimate on the infinite time region. Therefore, we provide an abstract frame to prove convergence. This method dosen't assume the existence of an equilibrium. It can be used to both the finite element methods and the difference methods, the explicit schemes and the implicit schemes.

The paper is outlined as follows: In 2, we give definitions of stability, compatibility and covergence of discrete schemes on the infinite time region. Especially, we obtain a theorem which states that the stability and compatibility imply convergence. In 3 and 4, we apply this theorem to the problem in [5] and obtain the similar results; to the problem in [6] and obtain the same results. In 5 we give an example whose continuous solution is periodical in time. It can't include in the frame of [3], [5] or [6]. But we can prove the long-time convergence of an explicit difference scheme from this theorem.

2. The Relation Between Stability and Convergence

Let V be a Banach space with norm $\|\cdot\|$. We consider the evolutionary equation such as:

$$\begin{aligned} &u_t - Au - f(u) = g(t), \\ &u(0) = u_0, \end{aligned} \tag{2.1}$$

here A and f are operators on a dense subset of V to V. Let $u(t) \in V$ be a solution of (2.1).

Let V_h be a finite dimensional Banach space with norm $\|\cdot\|_h$. It may or may not be the subspace of V. Let p_h be a operator from V to V_h. We denote the discretization of (2.1) as follows:

$$\begin{aligned} &B_{h,\tau}(u_{h,\tau}^{n+1}) = C_{h,\tau}(u_{h,\tau}^n) + \tau g_{h,\tau}^n, \quad n = 0, 1, 2, \cdots \\ &u_{h,\tau}^0 = u_{0,h}, \end{aligned} \tag{2.2}$$

where $g_{h,\tau}^n$, $u_{0,h} \in V_h$, $B_{h,\tau}$ and $C_{h,\tau}$ are operators on V_h, h is space step-size and τ is time step-size, $u_{h,\tau}^n$ is the approximation to $u(t_n)(t_n = n\tau)$.