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A GOLDSTEIN'S TYPE PROJECTION METHOD FOR A CLASS OF VARIANT VARIATIONAL INEQUALITIES*1)

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Abstract

Some optimization problems in mathematical programming can be translated to a variant variational inequality of the following form: Find a vector u^*, such that

$$Q(u^*) \in \Omega, \qquad \left(v - Q(u^*)\right)^T u^* \geq 0, \qquad \forall v \in \Omega.$$

This paper presents a simple iterative method for solving this class of variational inequalities. The method can be viewed as an extension of the Goldstein's projection method. Some results of preliminary numerical experiments are given to indicate its applications.

Key words: Variational inequality, Goldstein projection method.

1. Introduction

The classical variational inequality (VI) is to determine a vector u in a closed convex subset Ω of the n-dimentional Eucleaden space R^n such that

$$(v-u)^T F(u) \geq 0, \qquad \forall v \in \Omega, \tag{1}$$

where F is a mapping from R^n into itself. Let $\beta > 0$, since the early work of Eaves [3], it has been known that the variational inequality problem (VI) is equivalent to a projection equation

$$u = P_\Omega[u - \beta F(u)],$$

where $P_\Omega(\cdot)$ denotes the orthogonal projection map on Ω. In other words, to solve (VI) is equivalent to finding a zero point of the residue function

$$e(u,\beta) := u - P_\Omega[u - \beta F(u)].$$

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Among the existing methods (e.g., see [5–11,16,18–20]) for nonlinear variational inequality problems, the simplest is the Goldstein's projection method [6] which, starting with any $u^0 \in R^n$, iteratively updates u^{k+1} according to the formula

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^k)], \tag{2}$$

where β_k is a chosen positive stepsize. In contrast with Douglas–Rachford operator splitting method [2,12,13] for (VI), this projection method can be viewed as a simple explicit method, because u^{k+1} occurs only on the left-hand side of the equation in (2). Its convergence results can be found in [1,4] and [6].

In this paper, however, we consider a class of variant variational inequalities $(\mathrm{VI})_v$: Find an u, such that

$$Q(u) \in \Omega, \qquad (v - Q(u))^T u \geq 0, \qquad \forall v \in \Omega, \tag{3}$$

where $Q(u): R^n \to R^n$ is a function and $\Omega \subset R^n$ is a closed convex set. The existence results on such a problem have been investigated recently by Pang and Yao [17].

There are some methods in literature ([12,14–16]), which can be used for solving $(\mathrm{VI})_v$. However, our interest in this paper is to develop the simplest method–Goldstein's type projection method for solving the variant problem (3). Throughout this paper we assume that the solution set of $(\mathrm{VI})_v$, denoted by S^*, is nonempty and the projection on Ω is simple to carry out. The Eucliden norm in this paper will be denoted by $\|\cdot\|$.

2. Motivation and the Method

As the classical variational inequality is equivalent to

$$u = P_\Omega[u - \beta F(u)]$$

with a $\beta > 0$, it is easy to prove that the variant variational inequality (3) is equivalent to the following projection equation (PE)

$$Q(u) = P_\Omega[Q(u) - \beta u]. \tag{4}$$

Let

$$r(u, \beta) := \frac{1}{\beta}(Q(u) - P_\Omega[Q(u) - \beta u]) \tag{5}$$

denote the scaled *residue* of the (PE). Then we have

$$u \in S^* \iff r(u, \beta) = 0.$$

This tells us that to solve the variant variational inequality is equivalent to finding a zero point of $r(u, \beta)$. Note that the Goldstein's projection scheme (2) for (VI) can be viewed as

$$u^{k+1} = u^k - e(u^k, \beta_k). \tag{6}$$