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# A WAVELET METHOD FOR THE FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH CONVOLUTION KERNEL*1)

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**Abstract**

We study the Fredholm integro-differential equation

$$D_x^{2s}\sigma(x) + \int_{-\infty}^{+\infty} k(x-y)\sigma(y)dy = g(x)$$

by the wavelet method. Here $\sigma(x)$ is the unknown function to be found, $k(y)$ is a convolution kernel and $g(x)$ is a given function. Following the idea in [7], the equation is discretized with respect to two different wavelet bases. We then have two different linear systems. One of them is a Toeplitz-Hankel system of the form $(H_n + T_n)x = b$ where $T_n$ is a Toeplitz matrix and $H_n$ is a Hankel matrix. The other one is a system $(B_n + C_n)y = d$ with condition number $\kappa = O(1)$ after a diagonal scaling. By using the preconditioned conjugate gradient (PCG) method with the fast wavelet transform (FWT) and the fast iterative Toeplitz solver, we can solve the systems in $O(n \log n)$ operations.

*Key words*: Fredholm integro-differential equation, Kernel, Wavelet transform, Toeplitz matrix, Hankel matrix, Sobolev space, PCG method.

## 1. Introduction

In this paper, we study the Fredholm integro-differential equation

$$A(\sigma(x)) \equiv D_x^{2s}\sigma(x) + \int_{-\infty}^{+\infty} k(x-y)\sigma(y)dy = g(x) \tag{1}$$

by the wavelet method. The applications of the equation in image restoration could be found in [10]. For the history of numerical methods for the Fredholm integro-differential equations, we refer to [4]. Following the idea in [7], the equation is discretized with respect to two different orthonormal wavelet bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $L^2(R)$. The $\mathcal{B}_1$ comes from the father wavelet $\varphi(x)$ and the $\mathcal{B}_2$ comes from the mother wavelet $\psi(x)$. After discretizing of the equation with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ on a finite interval, we then have

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two different $n$-by-$n$ linear systems. One of them is a Toeplitz-Hankel system of the form

$$(H_n + T_n)x = b \tag{2}$$

where $T_n$ is a Toeplitz matrix and $H_n$ is a Hankel matrix. The other one is a system

$$(B_n + C_n)y = d \tag{3}$$

with condition number

$$\kappa(D_n^{-1/2}(B_n + C_n)D_n^{-1/2}) = O(1) \tag{4}$$

after a diagonal scaling $D_n$. The relation between $H_n + T_n$ and $B_n + C_n$ is $B_n + C_n = W_n(H_n + T_n)W_n^{-1}$ where $W_n$ is the wavelet transform matrix between $\mathcal{B}_1$ and $\mathcal{B}_2$.

We then solve (2) by solving its equivalent form (3) with $y = W_n x$ and $d = W_n b$. For solving (3), we use the PCG method with the diagonal preconditioner $D_n$. The condition number of the preconditioned system is, by (4),

$$\kappa(D_n^{-1}(B_n + C_n)) = \kappa(D_n^{-1/2}(B_n + C_n)D_n^{-1/2}) = O(1).$$

When the PCG method is applied to solve the preconditioned system, the convergence rate will be linear, see [5]. By using the FWT, see [2], and fast iterative Toeplitz solver, see [1] and [9], we can solve the system $(B_n + C_n)y = d$ and also $(H_n + T_n)x = b$ in $O(n \log n)$ operations.

## 2. Discretization of Fredholm Equation

The Fredholm integro-differential equation is given as follows, $A\sigma = g$, where $A$ is defined by (1), $g \in L^2(R)$ and $k(x-y) \in L^2(R)$ is symmetric and positive, i.e., $k(x-y) = k(y-x) > 0$. For solving the equation, we need to find $\sigma \in C_0^{2s}(R)$ such that (1) is to be satisfied. The equivalent variational form of (1) is: find $\sigma \in H_0^s(R)$ such that

$$B(\sigma, \mu) = F(\mu) \tag{5}$$

for $\forall \mu \in H_0^s(R)$. Here $B(\sigma, \mu) = B_0(\sigma, \mu) + B_1(\sigma, \mu)$ with

$$B_0(\sigma, \mu) = \int_{-\infty}^{+\infty} D_x^s \sigma(x) D_x^s \mu(x) dx,$$
$$B_1(\sigma, \mu) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k(x-y)\sigma(y)\mu(x) dy dx$$

and

$$F(\mu) = \int_{-\infty}^{+\infty} g(x)\mu(x) dx.$$

We assume that $B(\sigma, \mu)$ is a continuous elliptic bilinear form on $H_0^s(R) \times H_0^s(R)$, i.e., there exist two constants $\beta \geq \alpha > 0$, such that $\alpha \|\sigma\|_{H_0^s}^2 \leq B(\sigma, \sigma)$ and $B(\sigma, \mu) \leq \beta \|\sigma\|_{H_0^s} \|\mu\|_{H_0^s}$. For instance, when $s = 0$ (or $s = 1$) and $+\infty > C \geq k(x-y) \geq c > 0$, then obviously, $B(\sigma, \mu)$ is a continuous elliptic bilinear form on $L^2(R) \times L^2(R)$ (or $H_0^1(R) \times H_0^1(R)$).