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THE STABILITY OF THE θ-METHODS FOR DELAY DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the stability analysis of numerical methods for the solution of delay differential equations. We focus on the behaviour of three θ-methods in the solution of the linear test equation $u'(t) = A(t)u(t) + B(t)u(\tau(t))$ with $\tau(t)$ and $A(t), B(t)$ continuous matrix functions. The stability regions for the three θ-methods are determined.

Key words: Delay differential equations, Numerical solution, Stability, θ-methods.

1. Introduction

1.1. The three θ-methods

We deal with the numerical solution of the initial value problem:

$$\begin{cases} u'(t) = f(t, u(t), u(\tau(t))), & t > t_0, \\ u(t) = u_0(t), & t \leq t_0. \end{cases} \tag{1.1}$$

Here f, u_0, τ denote given functions with $\tau(t) \leq t$, whereas $u(t)$ is unknown (for $t > t_0$). With the so-called one-leg θ-method, linear θ-method and new θ-method, one can compute approximations u_n to $u(t)$ at the gridpoint $t_n = t_0 + nh$, where $h > 0$ denotes the stepsize and $n = 1, 2, 3, \cdots$.

The one-leg θ-method was considered in [1, 2, 3, 4]

$$\begin{aligned} &u_{n+1} = u_n + hf(\theta t_{n+1} + (1-\theta)t_n, \theta u_{n+1} + (1-\theta)u_n, \\ &u^h(\tau(\theta t_{n+1} + (1-\theta)t_n))), \quad n \geq 0 \end{aligned} \tag{1.2a}$$

where θ is a parameter, with $0 \leq \theta \leq 1$ specifying the method.

Further we define $u^h(t)$ as follows:

$$\begin{aligned} &u^h(t) = u_0(t), \quad t \leq t_0, \\ &u^h(t) = \frac{t_{n+1} - t}{h} u_n + \frac{t - t_n}{h} u_{n+1}, \quad t \in (t_n, t_{n+1}], \quad n \geq 0. \end{aligned}$$

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The linear θ-method to problem of type (1.1) gives rise to the following formula

$$u_{n+1} = u_n + h\{\theta f(t_{n+1}, u_{n+1}, u^h(\tau(t_{n+1}))) + (1-\theta) f(t_n, u_n, u^h(\tau(t_n)))\}, \quad n \geq 0, \qquad ((1.2b))$$

which was considered in [1, 2, 4–7].

Finally,we consider the new θ-method as follows:

$$\begin{aligned} u_{n+1} = u_n + hf(&\theta t_{n+1} + (1-\theta)t_n, \theta u_{n+1} + (1-\theta)u_n, \\ &\theta u^h(\tau(t_{n+1})) + (1-\theta)u^h(\tau(t_n))), \quad n \geq 0, \end{aligned} \qquad (1.2c)$$

which was considered in [1].

1.2. The test problem

Consider the test problem

$$\begin{cases} u'(t) = A(t)u(t) + B(t)u(\tau(t)), & t \geq t_0, \\ u(t) = u_0(t), & t \leq t_0. \end{cases} \qquad (1.3)$$

Here $A, B : [t_0, \infty) \to C^{d\times d}$ $(d \geq 1)$, $t - \tau(t) \geq \tau_0$ $(t \geq t_0)$, τ_0 is a positive constant, $u_0(t)$ is a known complex function for $t \leq t_0$.

Applying (1.2a), (1.2b), (1.2c) to (1.3) we have the following recurrence relations:

$$\begin{aligned} (I - \theta x(t_{n+\theta}))u_{n+1} =& (I + (1-\theta)x(t_{n+\theta}))u_n + \delta(t_{n+\theta})y(t_{n+\theta})u_{n-m(t_{n+\theta})+1} \\ &+ (1 - \delta(t_{n+\theta}))y(t_{n+\theta})u_{n-m(t_{n+\theta})}, \quad (n \geq m), \end{aligned} \qquad (1.4a)$$

Here

$$\begin{aligned} &\delta(t_{n+\theta}) = \frac{\tau(t_{n+\theta})}{h} - r(t_{n+\theta}), \\ &r(t_{n+\theta}) = \Big[\frac{\tau(t_{n+\theta})}{h}\Big], \quad \delta(t_{n+\theta}) \in [0, 1), \\ &m(t_{n+\theta}) = n - r(t_{n+\theta}), t_{n+\theta} = t_n + \theta h, \\ &x(t) = hA(t), \quad y(t) = hB(t). \end{aligned}$$

$$\begin{aligned} (I - \theta x(t_{n+1}))u_{n+1} =& (I + (1-\theta)x(t_n))u_n + \theta y(t_{n+1})(\delta(t_{n+1})u_{n+2-m(t_{n+1})} \\ &+ (1 - \delta(t_{n+1}))u_{n+1-m(t_{n+1})}) + (1-\theta)y(t_n)(\delta(t_n)u_{n+1-m(t_n)} \\ &+ (1 - \delta(t_n))u_{n-m(t_n)}), \quad n \geq m \end{aligned} \qquad (1.4b)$$

and

$$\begin{aligned} (I - \theta x(t_{n+\theta}))u_{n+1} =& (I + (1-\theta)x(t_{n+\theta}))u_n + \theta y(t_{n+\theta})(\delta(t_{n+1})u_{n+2-m(t_{n+1})} \\ &+ (1 - \delta(t_{n+1}))u_{n+1-m(t_{n+1})}) + (1-\theta)y(t_{n+\theta})(\delta(t_n)u_{n+1-m(t_n)} \\ &+ (1 - \delta(t_n))u_{n-m(t_n)}), \quad n \geq m. \end{aligned} \qquad (1.4c)$$

Here, $\delta(t) = \dfrac{\tau(t)}{h} - r(t), r(t) = \Big[\dfrac{\tau(t)}{h}\Big]$, $0 \leq \delta(t) < 1$, $m(t) = \dfrac{t}{h} - r(t)$.