Journal of Computational Mathematics, Vol.17, No.5, 1999, 457–462.

RELATIONS BETWEEN TWO SETS OF FUNCTIONS DEFINED BY THE TWO INTERRELATED ONE-SIDE LIPSCHITZ CONDITIONS*1)

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Abstract

In the theoretical study of numerical solution of stiff ODEs, it usually assumes that the righthand function $f(y)$ satisfy one-side Lipschitz condition

$$< f(y) - f(z), y - z > \leq \nu' \|y - z\|^2, f : \Omega \subseteq C^m \to C^m,$$

or another related one-side Lipschitz condition

$$[F(Y) - F(Z), Y - Z]_D \leq \nu'' \|Y - Z\|_D^2, F : \Omega^s \subseteq C^{ms} \to C^{ms},$$

this paper demonstrates that the difference of the two sets of all functions satisfying the above two conditions respectively is at most that $\nu' - \nu''$ only is constant independent of stiffness of function f.

Key words: Stiff ODEs, One-side Lipschitz condition, Logarithmic norm.

In the theoretical study of numerical solution of stiff ODEs, authors usually assume that the righthand function f of

$$y'(t) = f(y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad f : \Omega \subseteq C^m \to C^m, \tag{1}$$

satisfy the one-side Lipschitz condition[1,2,3]

$$< f(y) - f(z), y - z > \leq \nu \|y - z\|^2, \forall y, z \in \Omega, \tag{2}$$

* Received February 27, 1995.
1)Supported by the national natural science foundation.

however, in some cases(such as study of existence and uniqueness of the solution), the function f is assumed to satisfy another one-side Lipschitz condition

$$[F(Y) - F(Z), Y - Z]_D \leq \nu \|Y - Z\|_D^2, \tag{3}$$

where Ω is a convex domain in C^m, $Y = (y_1^T, y_2^T, \cdots, y_s^T)^T \in \Omega^s := \overbrace{\Omega \times \Omega \times \cdots \times \Omega}^{s \text{ times}}$, $F(Y)=(f^T(y_1), f^T(y_2),\cdots,f^T(y_s))^T$, $< \cdot, \cdot >$ is an inner-product in C^m, $\|\cdot\|$ is the corresponding norm, $D = (d_{ij})$is a s-by-s Hermite positive definite matrix, $[F(Y), Z]_D = \sum_{i,j=1}^s d_{ij} < f(y_i), z_j >$, $\|\cdot\|_D$ is the corresponding norm.

Definition:

$$\mathcal{F}_1(\nu) = \{f(y) \mid Re < f(y) - f(z), y - z > \leq \nu \|y - z\|^2,\ f'(y) \ is\ existed, \forall y, z \in \Omega\},$$

$$\mathcal{F}_2(\nu) = \{f(y) \mid Re[F(Y) - F(Z), Y - Z]_D \leq \nu \|Y - Z\|_D^2, f'(y) \ is\ existed, \forall Y, Z \in \Omega^s\},$$

where $f'(y)$ is a Frechet-derivative of $f(y)$ with respect to y. Up to date, there is no result for the relation of $\mathcal{F}_1(\nu)$ and $\mathcal{F}_2(\nu)$. The goal of this paper is to investigate this problem.

Theorem 1. *If D is a diagonally positive definite matrix, then*

$$\mathcal{F}_1(\nu) = \mathcal{F}_2(\nu).$$

Proof. For $\forall f(y) \in \mathcal{F}_2(\nu)$, it follows from the definition that

$$Re \sum_{i=1}^s d_{ii} < f(y_i) - f(z_i), y_i - z_i > = Re[F(Y) - F(Z), Y - Z]_D \leq \nu \|Y - Z\|_D^2, \tag{4}$$

if $f(y) \notin \mathcal{F}_1(\nu)$, then there exist $y, z \in \Omega$ such that

$$Re < f(y) - f(z), y - z > > \nu \|y - z\|^2.$$

Let $Y = (y^T, y^T, \cdots, y^T)^T$ and $Z = (z^T, z^T, \cdots, z^T)^T \in \Omega^s$, then

$$Re \sum_{i=1}^s d_{ii} < f(y) - f(z), y - z > > \nu \|Y - Z\|_D^2.$$

That is conflict with (4), so $\mathcal{F}_2(\nu) \subseteq \mathcal{F}_1(\nu)$. On the other hand, it is obvious that $\mathcal{F}_1(\nu) \subseteq \mathcal{F}_2(\nu)$. Therefore, $\mathcal{F}_1(\nu) = \mathcal{F}_2(\nu)$.

Theorem 2. *Assume that the D be a Hermite positive definite matrix and $f(y) = By + \hat{B}$ be a linear function, then $f \in \mathcal{F}_1(\nu) \iff f \in \mathcal{F}_2(\nu)$.*

Proof. For the inner-products $< \cdot, \cdot >$ and standard inner-product $(y, z) = y^* z$ in C^m, there exists a Hermite positive definite matrix Q such that

$$< y, z > = (y, Qz), \quad \forall y, z \in C^m.$$