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ON THE CONVERGENCE OF NONCONFORMING FINITE ELEMENT METHODS FOR THE 2ND ORDER ELLIPTIC PROBLEM WITH THE LOWEST REGULARITY*1)

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Abstract

The convergences ununiformly and uniformly are established for the nonconforming finite element methods for the second order elliptic problem with the lowest regularity, i.e., in the case that the solution $u \in H_0^1(\Omega)$ only.

Key words: Nonconforming finite element methods, Lowest regularity.

1. Introduction

The aim of this note is to establish the convergence of the nonconforming finite element methods for the second order elliptic problem with the lowest regularity. The proof of the convergence is not trivial, although the convergence results for the conforming finite element methods were known ([2], [3]).

Consider the following boundary value problem on a polygonal domain $\Omega \subset R^2$:

$$\begin{cases} Au = \sum_{i,j=1}^{2} -\partial_j(a_{ij}(x)\partial_i u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

We assume that the coefficients $a_{ij}(x) \in L^\infty(\Omega)$ and the A is uniformly elliptic on Ω, i.e., there exists a constant $\alpha > 0$ such that for all real vectors $\xi = (\xi_1, \xi_2)$ and all $x \in \Omega$

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{2} \xi_i. \tag{1.2}$$

The weak formulation of (1.1) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) \equiv \int_\Omega a_{ij}\partial_i u \partial_j v dx = \int_\Omega f v dx \equiv f(v), \quad \forall \ \ v \in H_0^1(\Omega). \tag{1.3}$$

It is well known that for any given $f \in H^{-1}(\Omega)$, there exists an unique solution $u \in H_0^1(\Omega)$ of the problem (1.3), by the Lax-Milgram Lemma, and the conforming finite element approximation u_h converges to u in $H^1(\Omega)$ space (c.f.[2]).

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We now consider the nonconforming finite element methods for the problem (1.3). For each $h \in (0,1)$, let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω, and V_h be a nonconforming finite element space with respect to the triangulation $\mathcal{T}_h$. In this case it should be noted that $V_h \not\subset H^1(\Omega)$, and assume that $f \in L^2(\Omega)$, while it can be assumed that $f \in H^{-1}(\Omega)$ for the conforming finite element methods, since the functional $f \in H^{-1}(\Omega)$ is defined on the space $H_0^1(\Omega)$ only. And it is also noted that the solution u of the problem (1.3) is, in general, in $H_0^1(\Omega)$ space only, in tha case of that $f \in L^2(\Omega)$, since that it is not known in general whether $u \in H^s(\Omega)$ for some $s > 1$ even if $f \in C^\infty(\Omega)$. Finally it is assumed that the element of the nonconforming finite element space V_h passes the generalized patch test, which is the necessary and sufficient condition, assuming the approximation holding, for the convergence of nonconforming finite element methods in the case of the solution u of the problem (1.3) smoother enough (c.f.[5]).

Then the nonconforming finite element approximation to (1.3) is: Find $u_h \in V_h$, such that

$$a_h(u_h, v_h) \equiv \sum_K \int_K a_{ij}\partial_i u_h \partial_j v_h dx = \int_\Omega f \cdot v_h dx \equiv f(v_h) \quad \forall \quad v_h \in V_h. \tag{1.4}$$

2. Convergence

Theorem 2.1. *Assume that the solution of the problem* (1.3) $u \in H_0^1(\Omega)$, $f \in L^2(\Omega)$, *the triangulation $\mathcal{T}_h$ of the polygonal Ω is quasi-uniform and satisfies the inverse hypothesis* (*c.f.*[2]), *and the nonconforming finite element space $V_h \not\subset H_0^1(\Omega)$ possessing the following property, for any given $\phi \in C_0^\infty$, there exists $C = Const. > 0$ independent of h, such that*

$$\Big|\sum_K \int_{\partial K} \partial_\nu \phi \cdot w_h ds\Big| \le Ch\|\phi\|_{2,\Omega} \cdot \|w_h\|_h, \quad \forall \quad w_h \in V_h, \tag{2.1}$$

where $K \in \mathcal{T}_h$ is the element with the edge $\partial K, \partial_\nu$ denotes the conormal derivative operator associated with the operator A in (1.1) *on ∂K, and*

$$\|w_h\|_h \equiv \Big\{\sum_K |w_h|_{1,K}^2\Big\}^{\frac{1}{2}}. \tag{2.2}$$

Then the solution of the problem (1.4) u_h *converges to the solution of the problem* (1.3) u *in the space $H^1(\Omega)$ as $h \longrightarrow 0$. Precisely, for any given $\epsilon > 0$, there exists $h_0 = h_0(\epsilon, u, f) > 0$, such that*

$$\|u - u_h\|_h < \epsilon, \quad as \quad 0 < h \le h_0. \tag{2.3}$$

Proof. (i) By the second Strang Lemma (c.f.[4])

$$\|u - u_h\|_h \le \Big\{\inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{E_h(u, w_h)}{\|w_h\|_h}\Big\}, \tag{2.4}$$