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# GENERALIZED DIFFERENCE METHODS ON ARBITRARY QUADRILATERAL NETWORKS*

Yong-hai Li Rong-hua Li
(*Institute of Mathematics, Jilin University, Changchun 130023, China*)

**Abstract**

This paper considers the generalized difference methods on arbitrary networks for Poisson equations. Convergence order estimates are proved based on some a priori estimates. A supporting numerical example is provided.

*Key words*: Quadrilateral elements, Dual grids, Bilinear functions, Generalized difference methods, Priori estimates, Error estimates.

## 1. Introduction

Consider the boundary value problem of the Poisson equation

$$\begin{cases} -\Delta u = f(x,y), & (x,y) \in \Omega & (1.1) \\ u = 0, & (x,y) \in \Gamma = \partial\Omega & (1.2) \end{cases}$$

where $\Omega$ is a convex polygon regon; $\Gamma = \partial\Omega$ the boundary of $\Omega$ and $f(x,y)$ a known function on $\Omega$.

The generalized difference methods on quadrilateral networks for elliptic equations are proposed in [11], where the convergence order estimates are given for rectangular networks. Quadrilateral networks are structured networks, the so called "finite volume method on structured networks" (cf. [7] - [9]), a popular method in computational fluid, is identical to the generalized difference method in [3](cf.[4] and [11]). The generalized difference methods have the same convergence orders as the corresponding finite element methods, but they require less computational expenses, and keep the mass conservation (cf. [5]). The aim of this paper is to provide a theory for the generalized difference method on arbitrary quadrilateral networks, and to obtain the optimal convergence order estimates. A generalized difference method with bilinear element is constructed in §2. Some a priori estimates are deduced in §3. §4 is devoted to the error order estimates. Finally, a numerical example is given in §5 to show the effectiveness of the method.

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## 2. Generalized Difference Methods

Let $\Omega$ be a convex polygonal region. Decompose $\Omega$ into the union of finite number of strictly convex and nonoverlapping quadrilateral elements.Two nodes are called adjacent if they are the endpoints of the same side of an element. The set of all the quadrilateral elements is denoted by $T_h$, where $h$ is the maximum length of all the sides.

Connect the midpoints of the opposite side of a quadrilateral element, and call the joint of the two connecting lines the averaging center. Now we construct the dual subdivision of $T_h$. Let $P$ be an inner node as in Fig.1; $\square PP_1P_2P_3$, $\square PP_3P_4P_5$, $\square PP_5P_6P_7$, $\square PP_7P_8P_1$ are the quadrilaterals with a common node $P$; and $Q_1, Q_2, Q_3, Q_4$ respectively are their averaging center. Let $M_1$, $M_2$, $M_3$, $M_4$ be the midpoints of $\overline{PP_1}, \overline{PP_3}, \overline{PP_5}, \overline{PP_7}$. Connect $M_1, Q_1$, $M_2, Q_2$, $M_3, Q_3$, $M_4, Q_4$, $M_1$, successively to obtain a polygonal region $K_P^*$ surrounding $P$, called a dual element. The set of all the dual elements is denoted by $T_h^*$, and called the dual subdivision (cf. [11] or [5]).

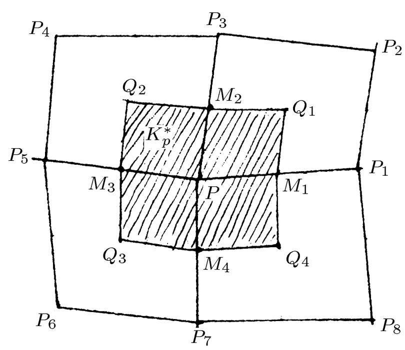


Fig. 1

Let $\bar{\Omega}_h$ be the set of nodes of $T_h$; $\overset{\circ}{\Omega}_h = \bar{\Omega}_h - \partial\Omega$ the set of the inner nodes; and $\Omega_h^*$ the set of nodes of the dual grid. Denote by $K_Q$ the quadrilateral element with averaging center $Q \in \Omega_h^*$, and by $S_Q, S_P^*$ the areas of the element $K_Q$ and the dual element $K_P^*$ respectively.

Suppose $T_h$ and $T_h^*$ are quasi-uniformly, that is, there exist constants $C_1, C_2 > 0$ independent of $h$, such that

$$C_1 h^2 \le S_Q \le h^2, \quad Q \in \Omega_h^* \qquad (2.1)_1$$

$$C_1 h^2 \le S_P^* \le C_2 h^2, \quad P \in \bar{\Omega}_h \qquad (2.1)_2$$

**Remark 1.** $(2.1)_2$ can be deduced from $(2.1)_1$ under the above assumptions on the dual grid.

In order to define the trial function space $U_h$, we take a unite square $\hat{K} = \hat{E}=$ [0,1]×[0,1] on $(\xi, \eta)$ plane as the reference element. For any convex quadrilateral