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ON A CELL ENTROPY INEQUALITY OF THE RELAXING SCHEMES FOR SCALAR CONSERVATION LAWS[*1)]

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Abstract

In this paper we study a cell entropy inequality for a class of the local relaxation approximation –The Relaxing Schemes for scalar conservation laws presented by Jin and Xin in [1], which implies convergence for the one-dimensional scalar case.

Key words: Hyperbolic conservation laws, the relaxing schemes, cell entropy inequality.

1. Introduction

In [1], Jin and Xin constructed a class of the local relaxation approximation-The Relaxing Schemes for systems of nonlinear conservation laws or single nonlinear conservation laws

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{d}\frac{\partial f_i(u)}{\partial x_i}=0, \tag{1.1}$$

with initial data $u(0,x)=u_0(x),\ x=(x_1,...,x_d)$, by using the idea of the local relaxation approximation[1-4].

The scheme is obtained in the following way: first a linear hyperbolic system with a stiff source term is constructed to approximate the original system (1.1) with a small dissipative correction; Then, the new linear hyperbolic system can be solved easily by underresolved stable numerical discretizations without using Riemann solvers spatially.

It is well known that the above Cauchy problem (1.1) may not always have a smooth global solution even if the initial data u_0 is smooth[1-3]. Thus, we consider its weak solution so that the problem (1.1) might have a global solution allowing discontinuities(e.g. shock wave etc.). Moreover, the entropy condition must be imposed in order to single out a physically relevant solution(also called the entropy solution)[7-9].

For the numerical approximation method of the equation (1.1), the numerical entropy condition(e.g. the proper cell entropy inequality) must be imposed on it in order that the numerical solution can converges to the entropy solution of the above problem. However, the entropy conditions seems difficulty to prove for high-order finite difference schemes[10-11].

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Here, we will study the entropy condition for the semidiscrete relaxing schemes for scalar conservation laws with general flux. The paper is organized as follows. In section 2, we simply recall the relaxing system with a stiff source term, constructed by Jin et al. to approximate the equation (1.1). In section 3, we establish the relation between the entropy pair for the relaxing system and the entropy pair for the system (1.1). In section 4, we discuss the entropy conditions for the semidiscrete first order upwind relaxing scheme and second order MUSCL-type relaxing scheme.

2. Preliminaries

In this section, we will review the relaxing system with a stiff source, constructed by Jin and Xin. to approximate the equation (1.1). In the following, we will only consider single scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.1}$$

with initial data

$$u(0,x) = u_0(x). \tag{2.2}$$

As in [1], a linear system with a stiff source term (hereafter called the **relaxing system**) can be constructed as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + a\dfrac{\partial u}{\partial x} = -\dfrac{1}{\epsilon}(v - f(u)), \end{cases} \tag{2.3}$$

where the small positive parameter ϵ is the relaxation rate, and a is a positive constant satisfying

$$| f'(u) | \leq \sqrt{a}, \text{ for all } u \in R. \tag{2.4}$$

Remark. Here, we can consider the more general $a(x,t)$ instead of the above constant a. The results in this paper are not limited by the above constant a.

In the small relaxation limit $\epsilon \to 0^+$, the relaxing system(2.3) can be approximated to leading order by the following *relaxed* equations

$$v = f(u), \tag{2.5}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.6}$$

The state satisfying (2.5) is called the **local equilibrium**. By the Chapman-Enskog expansion[12], we can derive the following first order approximation to (2.3)

$$v = f(u) - \epsilon\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}, \tag{2.7}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon\frac{\partial}{\partial x}\Big(\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}\Big), \tag{2.8}$$

It is clear that the above second equation (2.8) is dissipative under condition (2.4) (which is referred to as the *subcharacteristic condition* by T.-P. Liu in [2]). Here, we can choose the special initial condition for the relaxing system (2.3) as follows:

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x) \equiv f(u_0(x)). \end{aligned} \tag{2.9}$$