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THE GPL-STABILITY OF RUNGE-KUTTA METHODS FOR DELAY DIFFERENTIAL SYSTEMS*[1)]

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Abstract

This paper deals with the GPL-stability of the Implicit Runge-Kutta methods for the numerical solutions of the systems of delay differential equations. We focus on the stability behaviour of the Implicit Runge-Kutta(IRK) methods in the solutions of the following test systems with a delay term

$$\begin{aligned} y'(t) &= Ly(t) + My(t-\tau), \quad t \geq 0, \\ y(t) &= \Phi(t), \quad t \leq 0, \end{aligned}$$

where L, M are $N \times N$ complex matrices,$\tau > 0$,$\Phi(t)$ is a given vector function. We shall show that the IRK methods is GPL-stable if and only if it is L-stable,when we use the IRK methods to the test systems above.

Key words: Delay differential equation, Implicit Runge-Kutta methods, GPL-stability.

1. Introduction

Before dealing with the numerical stabililty analysis of the IRK methods for systems of DDEs,we consider the following initial value problem

$$y'(t) = f(t, y(t)), \quad t > t_0, \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

where f is a given function and $y(t)$ is unknown for $t > t_0$.

For the initial problem (1)-(2),consider an Implicit Runge-Kutta method,

$$K_{n,i} = hf(t_n + c_i h, y_n + \sum_{j=1}^{v} a_{ij} K_{n,j}), \quad i = 1, 2, \ldots, v, \tag{3}$$

$$y_{n+1} = y_n + \sum_{i=1}^{v} b_i K_{n,i}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $\sum_{i=1}^{v} b_i = 1, c_i = \sum_{j=1}^{v} a_{ij}, 1 \leq i \leq v, y_n \sim y(t_n), t_n = t_0 + nh$ and $h > 0$ is a stepsize.

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When we want to analyze the numerical stability of the IRK methods,we focus on the stability behaviour of the IRK methods with respect to the following linear test equations

$$y'(t) = \lambda y(t), \quad Re(\lambda) < 0, \tag{5}$$

$$y(t_0) = y_0. \tag{6}$$

We get the numerical recurrence formula,(see [7])

$$y_{n+1} = r(\bar{h})y_n, \quad n \geq 0, \tag{7}$$

$$r(\bar{h}) = 1 + \bar{h}\mathrm{b}^T(I - \bar{h}A)^{-1}e = \frac{det[I - \bar{h}(A - e\mathrm{b}^T)]}{det[I - \bar{h}A]}, \text{ if } det[I - \bar{h}A] \neq 0. \tag{8}$$

Definition 1.1. (see [7]) *Let $R(q)$ be a function of q.*

(a) *If $Re(q) < 0 \Longrightarrow |R(q)| < 1$,then we say $R(q)$ is A-acceptable;*

(b) *If $q < 0 \Longrightarrow |R(q)| < 1$,then we say $R(q)$ is A_0-acceptable;*

(c) *If $R(q)$ is A-acceptable and $\lim_{Re(q)\to-\infty} |R(q)| = 0$, then we say $R(q)$ is L-acceptable.*

From Definition 1.1, we have the following statements. For the Implicit Runge-Kutta methods (3)-(4),
(1) it is A-stable if and only if $r(\bar{h})$ is A-acceptable;
(2) it is L-stable if and only if $r(\bar{h})$ is L-acceptable.

2. The GPL-Stability of the IRK Methods

For the following systems of delay differential equations

$$y'(t) = Ly(t) + My(t - \tau), \quad t \geq 0, \tag{9}$$

$$y(t) = \Phi(t), \quad t \leq 0, \tag{10}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_N(t))^T$,$L$ and M are constant complex $N\times N$ matrices,$\tau > 0$,$\Phi(t)$ denotes a given vector value function and $y(t)$ is unknown for $t > 0$.

We consider the exponential solutions of (9)-(10) in the form

$$y(t) = \xi \cdot e^{\zeta t}, \quad \xi \in C^N. \tag{11}$$

We have

Lemma 2.1. (see [5]) *The systems (9) has nonzero exponential solutions if and only if*

$$det[\zeta I - L - Me^{-\zeta\tau}] = 0. \tag{12}$$

The equation (12) is called the characteristic equation of (9),and (9) is asymptotically stable if and only if every root ζ of (12) satisfies $Re(\zeta) < 0$.

Lemma 2.2. (see [5]) *Assume that the coefficients of (9) satisfy*

$$\eta(L) = \frac{1}{2}\lambda_{\max}(L + L^*) < 0, \tag{13}$$

$$||M|| < -\eta(L), \tag{14}$$

then all roots of the equation (12) have negative real parts and the systems of (9) is asymptotically stable,i.e.,$\lim_{t\to-\infty} y(t) = 0$.