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THE OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR HARMONIC EQUATION OVER EXTERIOR THREE-DIMENSIONAL DOMAIN*1)

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Abstract

In this paper, the overlapping domain decomposition method, which is based on the natural boundary reduction[1] and first suggested in [2], is applied to slove the exterior boundary value problem of harmonic equation over three-dimensional domain. The convergence and error estimates both for the continuous case and the discrete case are given. The contraction factor for the exterior spherical domain is also discussed. Moreover, numerical results are given which show that the accuracy and the convergence are in accord with the theoretical analyses.

Key words: DDM, Exterior harmonic problem, Natural boundary reduction.

1. Introduction

Many scientific and engineering problems can be reduced to exterior boundary value problems of partial differential equations. Although the numerical methods to solve boundary value problems, such as the finite element method and the finite difference method, are very effective on bounded domains, yet we often find it difficult to use them to deal with unbounded problems. The boundary reduction is a forceful means to solve some problems over unbounded domains. Among many boundary reductions, the natural boundary reduction founded by K. Feng and D.H. Yu has some distinctive advantages[1]. However, it also has its own limitation. How to Combine the natural boundary element method with the traditional finite element method effectively to solve problems over unbounded domains is an interesting and worthy work. Up to now, there have been some advances in this field in the two-dimensional cases[1−4]. Our goal here will be to extend this work to three-dimensional field.

Consider the following Dirichlet exterior boundary value problem

$$\begin{cases} -\triangle u = 0, & \text{in } \Omega^c, \\ u = g_0, & \text{on } \Sigma_0, \end{cases} \tag{1}$$

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where Ω is a bounded domain in R^3 with regular boundary $\partial\Omega = \Sigma_0$ and Ω^c denotes $R^3\backslash\overline{\Omega}$. In order to assure the existence and uniqueness of the solution of (1), the assumption that u vanishes at infinity is needed. The corresponding variational form of (1) is

$$\begin{cases} \text{Find } w = u - \widetilde{u} \in \overset{\circ}{W}{}_0^1(\Omega^c), \text{ such that} \\ D_{\Omega^c}(w,\ v) = -D_{\Omega^c}(\widetilde{u},\ v),\ \forall\ v \in \overset{\circ}{W}{}_0^1(\Omega^c), \end{cases} \tag{2}$$

where

$$D_{\Omega^c}(u,\ v) = \int_{\Omega^c} \nabla u \bullet \nabla v dxdydz,$$
$$\overset{\circ}{W}{}_0^1(\Omega^c) = \{v \in W_0^1(\Omega^c) \mid v|_{\Sigma_0} = 0\},$$
$$W_0^1(\Omega^c) = \{v \mid \frac{v}{\sqrt{1+x^2+y^2+z^2}},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial z} \in L^2(\Omega^c)\},$$

$\widetilde{u} \in W_0^1(\Omega^c)$ has bounded support and $\widetilde{u}\,|_{\Sigma_0} = g_0$. $|u|_1 = \sqrt{D_{\Omega^c}(u,\ u)}$ is an equivalent norm of $\overset{\circ}{W}{}_0^1(\Omega^c)$. If $g_0 \in H^{\frac{1}{2}}(\Sigma_0)$, then there exists $\widetilde{u}$ such that the solution of (2) exists and is uniquely determined[5]. Particularly, if Ω is a spherical domain with radius R whose centre is the origin of coordinates and g_0 is continuous on Σ_0, the solution of (2) is given by the following Poisson integral formula

$$u(r,\ \theta,\ \varphi) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{(r^2-R^2)\sin\theta' g_0(\theta',\ \varphi')}{(R^2+r^2-2Rr\cos\gamma)^{3/2}} d\theta' d\varphi',\ r > R, \tag{3}$$

where $(r,\ \theta,\ \varphi)$ denote the spherical coordinates and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi').$$

By using (3), we will develop an overlapping domain decomposition algorithm for general unbounded domain Ω^c.

2. Schwarz Alternating Algorithm and Its Convergence

Make a sphere $\Sigma_2 = \{(r,\ \theta,\ \varphi)|r = R\}$ for an appropriate R such that Σ_0 is surrounded by Σ_2. Then make a regular closed surface Σ_1 which surrounds Σ_2. Let Ω_1 denote the annular domain between Σ_0 and Σ_1. Define Ω_2 as the exterior unbounded domain with boundary Σ_2. Then the original problem is turned into two subproblems over subdomains Ω_1 and Ω_2. Construct the following Schwarz alternating algorithm

$$\begin{cases} -\Delta u_1^{(2k)} = 0, & \text{in } \Omega_1, \\ u_1^{(2k)} = g_0, & \text{on } \Sigma_0, \\ u_1^{(2k)} = u_2^{(2k-1)}, & \text{on } \Sigma_1, \end{cases} \tag{4}$$

and

$$\begin{cases} -\Delta u_2^{(2k+1)} = 0, & \text{in } \Omega_2, \\ u_2^{(2k+1)} = u_1^{(2k)}, & \text{on } \Sigma_2, \end{cases} \tag{5}$$