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JACOBI SPECTRAL APPROXIMATIONS TO DIFFERENTIAL EQUATIONS ON THE HALF LINE*

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Abstract

Some Jacobi approximations are investigated, which are used for numerical solutions of differential equations on the half line. The stability and convergence of the proposed schemes are proved. The main idea and techniques in this paper are also applicable to other problems on unbounded domains.

Key words: Jacobi spectral approximations, Differential equations on the half line, Stability and convergence.

1. Introduction

Many physical problems are set on unbounded domains. Some conditions at infinity are given by certain asymptotic behaviors of the solutions. For numerical simulations, we often restrict calculations to bounded domains, and impose certain artificial boundary conditions, which usually cause numerical errors . If we use spectral methods associated with orthogonal systems on unbounded domains, the mentioned troubles might be remedied. Maday, Pernaud-Thomas and Vandeven [1], Coulaud, Funaro and Kavian [2], and Funaro [3] considered Laguerre spectral approximations for linear problems on the half line. Iranzo and Falquès [4] provided some Laguerre pseudospectral schemes and Laguerre Tau schemes. Mavriplis [5] studied Laguerre spectral element method. On the other hand, Funaro and Kavian [6] proved the convergence of spectral and pseudospectral methods using Hermite functions for some linear problems on the whole line. While Guo [7] developed a spectral method using Hermite polynomials for the Burgers equation on the whole line, and Guo and Shen [8] proposed some spectral schemes using Laguerre polynomials for the Benjamin-Bona-Mahony equation and the Burgers equation on the half line. They also provided an efficient algorithm, and proved the stability and convergence of the proposed schemes . However all of these algorithms need certain quadratures on unbounded domains, which introduce errors and so weaken the merit of spectral approximations. Another approach is to use rational basis functions, see Christov [9], Boyd [10], and Iranzo and Falquès [4]. Recently, Guo [11] developed another method in which differential equations on the whole line are changed to certain problems on a finite interval. Since the coefficients of the resulting equations degenerate at both extreme points, a specific Gegenbauer approximation was

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used for their numerical solutions. Guo [11] also proved the stability and convergence of the proposed schemes .

This paper is devoted to Jacobi spectral method for differential equations on the half line. We change differential equations on the half line to certain problems on a finite interval. Since the coefficients of the resulting equations degenerate only at one of extreme points, it is not reasonable to approximate them by Gegenbauer polynomials. A natural choice is to use certain unsymmetric Jacobi approximations. In the next section, we introduce some weighted spaces and related unsymmetric Jacobi polynomials. Several weighted Poincaré inequalities and weighted inverse inequalities are obtained. In section 3, we focus on some orthogonal projections and derive the corresponding approximation results. All results in these two sections play important roles in the error analysis. The final section is for the application of unsymmetric Jacobi approximations. In particular, we take the Burgers equation as an example to show how to deal with nonlinear problems. The stability and convergence of the proposed schemes are proved. The main idea and techniques used in this paper are also useful for other problems on the half line. It is not difficult to generalize the main results in this paper to those on multiple-dimensional unbounded domains.

2. Some Jacobi Polynomials and Weighted Inequalities

Let $\Lambda = \{x \mid -1 < x < 1\}$ and $\chi(x)$ be a certain weight function in the usual sense. For $1 \le p \le \infty$, let

$$L^p_\chi(\Lambda) = \{v \mid v \text{ is measurable and } ||v||_{L^p_\chi} < \infty\}$$

where

$$||v||_{L^p_\chi} = \begin{cases} \left(\int_\Lambda |v(x)|^p \chi(x)dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Lambda} |v(x)|, & p = \infty. \end{cases}$$

In particular, $L^2_\chi(\Lambda)$ is a Hilbert space with the following inner product and norm

$$(u,v)_\chi = \int_\Lambda u(x)v(x)\chi(x)dx, \qquad ||v||_\chi = (v,v)^{\frac{1}{2}}_\chi.$$

Further, let $\partial_x v(x) = \frac{\partial}{\partial x} v(x)$, and for any non-negative integer m, define

$$H^m_\chi(\Lambda) = \{v \mid \partial^k_x v \in L^2_\chi(\Lambda), 0 \le k \le m\}$$

equipped with the inner product, semi-norm and norm as follows

$$(u,v)_{m,\chi} = \sum_{k=0}^{m} (\partial^k_x u, \partial^k_x v)_\chi,$$

$$|v|_{m,\chi} = ||\partial^m_x v||_\chi, \qquad ||v||_{m,\chi} = (v,v)^{\frac{1}{2}}_{m,\chi}.$$

For any real $r \ge 0$, we define the space $H^r_\chi(\Lambda)$ with the norm $||v||_{r,\chi}$ by space interpolation as in Adams [12]. Let $\mathcal{D}(\Lambda)$ be the set of all infinitely differentiable functions with compact support in Λ, and $H^r_{0,\chi}(\Lambda)$ be the closure of $\mathcal{D}(\Lambda)$ in $H^r_\chi(\Lambda)$. For $\chi(x) \equiv 1$, we denote the spaces $H^r_\chi(\Lambda)$ and $H^r_{0,\chi}(\Lambda)$ by $H^r(\Lambda)$ and $H^r_0(\Lambda)$. Their semi-norm and