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A FINITE DIFFERENCE SCHEME FOR THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS*

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Abstract

A finite difference scheme for the generalized nonlinear Schrödinger equation with variable coefficients is developed. The scheme is shown to satisfy two conservation laws. Numerical results show that the scheme is accurate and efficient.

Key words: Finite difference scheme, Schrödinger equation, Discrete energy method.

1. Generalized Nonlinear Schrödinger Equation

The Schrödinger equation has been extensively used in physics research, particularly in the modeling of nonlinear dispersion waves [8]. Numerical methods for solving the Schrödinger equation have been discussed in the literature. In this article, we consider a generalized nonlinear Schrödinger equation with variable coefficients

$$i\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(A(x)\frac{\partial u}{\partial x}) + iF(t)u + B(x)\,|u|^{p-1}\,u = 0, \quad i^2 = -1, \quad p > 1, \tag{1}$$

where $u(x,0) = \phi(x)$. The coefficients $A(x), F(t)$ and $B(x)$ are real functions with $A(x) > 0$, and $\phi(x)$ a sufficiently smooth function which vanishes for sufficiently large $|x|$. The solution $u(x,t)$ is a complex-valued function defined over the whole real line R. The above equation is a generalized case of those equations described in the literature [2,3,7]. In [2,3] the authors considered that the coefficient of u (which is the third term on the left-hand side of the Eq. (1)) was a real function rather than a complex function $iF(t)$. We find in the next text that the conservation laws for these two cases are different. In [7] the authors considered that the coefficient of u was a constant complex number iv rather than a complex function $iF(t)$. When $F(t) = v > 0$, there is a strong dissipative term resulting in amplitude decay of the soliton for the problem of propagation of a single soliton. However, the obtained numerical scheme produced small ripples for solving the propagation of a soliton [7]. Authors in [7] pointed out

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that as a result of the ripple effect in the numerical solution other methods should be explored.

To derive two conservation laws for Eq. (1), we first let $u = we^{-\int F(t)dt}$ to eliminate the term $iF(t)u$. For convenience, we assume here that $\int F(t)dt = 0$ when $F(t) = 0$. As such, one obtains

$$i\frac{\partial w}{\partial t} - \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x}) + B(x)e^{-(p-1)\int F(t)dt}\,|w|^{p-1}\,w = 0. \tag{2}$$

Multiplying Eq. (2) by $\bar{w}$ (which is the conjugate of w), integrating over the whole real line and taking the imaginary part, one obtains

$$\mathrm{Im}\int_R \left(i\frac{\partial w}{\partial t}\bar{w} - \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x})\bar{w} + B(x)e^{-(p-1)\int F(t)dt}\,|w|^{p+1}\right)dx = 0.$$

Since

$$\mathrm{Re}(\bar{w}\frac{\partial w}{\partial t}) = \frac{1}{2}(\bar{w}\frac{\partial w}{\partial t} + w\overline{\frac{\partial w}{\partial t}}) = \frac{1}{2}\frac{\partial |w|^2}{\partial t}$$

and

$$\begin{aligned}\int_R \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x})\bar{w}dx &= \int_R \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x}\bar{w})dx - \int_R A(x)\frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x}dx \\ &= -\int_R A(x)\frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x}dx \\ &= -\int_R A(x)\left|\frac{\partial w}{\partial x}\right|^2 dx\end{aligned}$$

then $\frac{d}{dt}\int_R |w|^2\,dx = 0$ from the imaginary part. Here, w is zero in the limit at $\pm\infty$ since the initial condition $\phi(x)$ is a sufficiently smooth function and vanishes for sufficiently large $|x|$. Replacing w by $ue^{\int F(t)dt}$, we obtain $\frac{d}{dt}\left(e^{2\int F(t)dt}\int_R |u|^2\,dx\right) = 0$. Hence, the first conservation law can be written as follows:

$$\int_R |u(x,t)|^2\,dx = \int_R |\phi(x)|^2\,dx \cdot e^{-2\int F(t)dt}. \tag{3}$$

It can be seen from Eq. (3) that the first conservation law is the same as that obtained in the literature [7,8,10,11,12] if $F(t) = 0$ or constant v.

We now multiply Eq. (1) by $\frac{\partial \bar{w}}{\partial t}$, integrate over R and take the real part to obtain

$$\mathrm{Re}\int_R \left(i\left|\frac{\partial w}{\partial t}\right|^2 - \frac{\partial}{\partial x}(A(x)\frac{\partial w}{\partial x})\frac{\partial \bar{w}}{\partial t} + B(x)e^{-(p-1)\int F(t)dt}\,|w|^{p-1}\,w\frac{\partial \bar{w}}{\partial t}\right)dx = 0. \tag{4}$$