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AN EXPLICIT PSEUDO–SPECTRAL SCHEME WHIT ALMOST UNCONDITIONAL STABILITY FOR THE CAHN–HILLIARD EQUATION*1)

Bai-nian Lu
(*Department of Mathematics, Shaanxi Normal University, Xi'an 710062, China*)

Rui-feng Zhang
(*Department of Mathematics, Kaifeng Normal Collage, Kaifeng, China*)

Abstract

In this paper, an explicit fully discrete three-level pseudo-spectral scheme with almost unconditional stability for the Cahn–Hilliard equation is proposed. Stability and convergence of the scheme are proved by Sobolev's inequalities and the bounded extensive method of the nonlinear function (B.N. Lu[4] (1995)). The scheme possesses the almost same stable condition and convergent accuracy as the Creak-Nicloson scheme but it is an explicit scheme. Thus the iterative method to solve nonlinear algebraic system is avoided. Moreover, the linear stability of the critical point u_0 is investigated and the linear dispersive relation is obtained. Finally, the numerical results are supplied, which checks the theoretical results.

Key words: Cahn-Hilliard equation, Pseudo-spectral scheme, Almost unconditional stability, Linear stability for criticlal points, Numerical experiments.

1. Introduction

In this paper we consider a class of the nonlinear Cahn–Hilliard equation with periodic initial-value problem:

$$\begin{cases} u_t = M\Delta(\phi(u) - \gamma\Delta u), & (x,t) \in R \times J & (1.1) \\ u(x,0) = u_0(x), & x \in R & (1.2) \\ u(x+2\pi, t) = u(x,t). & (x,t) \in R \times J & (1.3) \end{cases}$$

where $M > 0$ is the mobility (assumed to be a constant) and $\gamma > 0$ is a phenomena logical constant modeling the effect of interfacial energy. The Laplace operator is denoted by Δ, $\phi(u) = \psi'(u)$, $\psi(u) = \frac{1}{4}(u^2 - \beta^2)^2$ is called the homogeneous free energy. $\phi(\cdot)$ is the real function; $u_0(x)$ and $u(x,t)$ are the given and unknown real functions defined on R and $R \times J$, 2π–periodic with respect to x, respectively. $J = [0,T](T > 0)$. R is the real line.

Theoretical results about the existence uniqueness and regularity for (1) can be found in [1]. Numerical approximations of (1) based on the finite element method[2], the finite difference method[3] have also been considered.

In this paper, we devote a three-level explicit pseudo–spectral with almost same stable condition and same accuracy as Creak-Nicloson implicit scheme. We prove its

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convergence and stability by using the bounded extensive method of the nonlinear function[4]. Therefore we avoid quite difficult a priori estimates. We don't need to solve nonlinear algebraic system.

Throughout this paper, the c will be used to indicate generic constants, dependent of constant M, γ, T, function ϕ, u_0, and so on.

2. The Pseudo–Spectral Scheme

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm of $L^2(I)$ defined by

$$(u,v)=\int_I u\cdot\bar{v}dx,\quad \|u\|^2=(u,u)$$

where $I=[0,2\pi)$. Moreover, we define the Sobolev norm and seminorm:

$$\|u\|_s^2=\sum_{j=0}^{s}\Big\|\frac{\partial^j u}{\partial x^j}\Big\|^2,\quad |u|_j^2=\Big\|\frac{\partial^j u}{\partial x^j}\Big\|^2.$$

The definition of periodical Sobolev space $H_p^s(I)$ may be found in [4, 6–7].

The Fourier modes $\chi_j(x)=\dfrac{1}{\sqrt{2\pi}}e^{ijx}$, $j=0,\pm1,\pm2,\cdots$ are an orthogonal Hilbert basis of $L_p^2(I)$. For any positive even integer N we set

$$S_N=\text{Span}\Big\{\chi_j(x):-\frac{N}{2}\le j\le\frac{N}{2}-1\Big\}$$

and we denote by P_N the orthogonal project of $H_p^s(I)$ upon S_N.

Let K be a positive integer and $k=T/K$ be the time–step length. The notation u_N^n is used to denote the approximation of u_N at $t=nk$.

We define the following difference quotients:

$$u_{N\hat{t}}^n=\frac{u_N^{n+1}-u_N^{n-1}}{2k};\qquad u_N^{n+\frac12}=\frac12(u_N^{n+1}+u_N^{n-1}).$$

Let $h=2\pi/N$ be the space–step length and $x_j=jh$ $(0<j\le N)$. The discrete inner product in the interval I is define by

$$(u,v)_h=h\sum_{j=1}^{N}u(x_j)\bar{v}(x_j),\quad \|u\|_h^2=(u,u)_h.$$

The approximation u_N^n to u_N at $t=nk$ given by the pseudo–spectral method is defined by the equations:

$$\begin{cases}(u_{N\hat{t}}^n,\chi)_h=M(\phi(u_N^n),\Delta\chi)_h-M\gamma(\Delta u_N^{n+\frac12},\Delta\chi)_h, & \forall\chi\in S_N, \quad (2.1)\\ \dfrac1k(u_N^1-u_N^0,\chi)_h=M(\phi(u_N^0),\Delta\chi)_h-M\gamma(\Delta u_N^0,\Delta\chi)_h, & \forall\chi\in S_N, \quad (2.2)\\ (u_N^0,\chi)_h=(u_0,\chi)_h, & \forall\chi\in S_N. \quad (2.3)\end{cases}$$

Lemma 1.[5] *For any $f,g\in C(\bar{I})$.*

$$(I_Nf,I_Ng)_h=(I_Nf,I_Ng)=(f,g)_h,$$

where I_N is the interpolative operator defined by $I_Nf\in S_N$ and $I_Nf(x_j)=f(x_j)$.

Lemma 2.[5] *Assume that $v\in H_p^s(I)$, for any $s\ge\mu\ge0$, then there exists a positive constant c, independent of v and N, such that $\|v-I_Nv\|_\mu\le cN^{\mu-s}|v|_s$.*

Lemma 3.[5] *Let $\sigma\ge\mu\ge0$, for any $v\in S_N$, then $|v|_\sigma\le(N/2)^{\sigma-\mu}|v|_\mu$, $\sigma\ge1/2$.*

By lemma 1, we can obtain the following result: