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THE BLOSSOM APPROACH TO THE DIMENSION OF THE BIVARIATE SPLINE SPACE*[1)]

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Abstract

The dimension of the bivariate spline space $S_n^r(\Delta)$ may depend on geometric properties of triangulation Δ, in particular if n is not much bigger than r. In the paper, the blossom approach to the dimension count is outlined. It leads to the symbolic algorithm that gives the answer if a triangulation is singular or not. The approach is demonstrated on the case of Morgan-Scott partition and twice differentiable splines.

Key words: Bivariate spline space, Blossom, Dimension.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a closed simply connected polygonal region, and

$$\Delta := \{\Omega_i\}_{i=1}^t, \ \Omega = \bigcup_{i=1}^t \Omega_i$$

its regular triangulation, i.e. the triangles

$$\Omega_i, \Omega_j, \ i \neq j,$$

can have in common only a vertex or a whole edge. Let V denote the set of inner vertices, E the set of inner edges, and $\overline{E}$ the set of all edges of Δ. Put

$$m_V := |V|, \quad m_E := |E|.$$

The planar graph $G := (V, \overline{E})$ clearly describes Δ. However, it's sometimes useful to consider also the dual planar graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where vertices $i \in \mathcal{V}$ correspond to triangles Ω_i, and $e = (i,j) \in \mathcal{E}$ iff Ω_i,Ω_j share a common edge. Note $|\mathcal{V}| = t$, $|\mathcal{E}| = m_E$, and there is one-to-one correspondance between E and $\mathcal{E}$. So we shall not make any difference between $e = (i,j) \in \mathcal{E}$, and the common edge of Ω_i, Ω_j if not neccessary. In particular, $||e||$ will denote the length of the common edge of the corresponding

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triangles, direction of e will be the direction of this common edge etc. There is another simple relation between G and $\mathcal{G}$, a one to one correspondence between the vertices $v \in V$, and elementary cycles in $\mathcal{G}$,

$$Y_v = ((i_1, i_2), (i_2, i_3), \dots, (i_d, i_{d+1})),\ i_{d+1} = i_1,\ (i_j, i_{j+1}) \in \mathcal{E},$$

the boundaries of facets. Here, d denotes the degree of v. The cycle Y_v describes the connection between the triangles that meet at inner vertex v. $\mathcal{G}$ is a planar cubic graph with m_V elementary cycles. By Euler's equation,

$$m_E - m_V = t - 1. \tag{1.1}$$

Let $\pi_n(\mathbb{R}^2)$ denote the space of polynomial functions of total degree $\leq n$, and let

$$S_n^r(\Delta) := \{f | f|_{\Omega_i} \in \pi_n(\mathbb{R}^2)\} \cap C^r(\Omega)$$

denote the spline space over a regular triangulation Δ. Quite clearly

$$\dim \pi_n(\mathbb{R}^2) = \binom{n+2}{2}, \tag{1.2}$$

but the dimension of $S_n^r(\Delta)$ may be hard to determine since it might depend on the geometric properties of the triangulation. One can find a lower bound ([9], [10]) as

$$\begin{aligned} \dim S_n^r(\Delta) \quad \geq \quad & \Phi_n^r(\Delta) := \binom{n+2}{2} + \binom{n-r+1}{2} m_E \\ & - \left(\binom{n+2}{2} - \binom{r+2}{2} \right) m_V + \sum_{i=1}^{m_V} \sigma_i, \\ & \sigma_i := \sum_{j=1}^{n-r} (r + j + 1 - j n_i)_+,\ i = 1, 2, \dots, m_V. \end{aligned} \tag{1.3}$$

Here n_i denotes the number of edges with different slopes at inner vertex $v_i \in V$. A similar expression for the upper bound can be established also. Particular partitions show that the lower bound is often very close to actual dimension of the spline space. As an example, in [3] they can differ only by 1. Also, if n is large enough, i.e. $n \geq 3r+2$, and (1.3) actually gives the required dimension ([4]).

In this paper we will tackle the spline space dimension problem by relations, derived from the blossoming formulation of the continuity conditions. In order to proceed let us recall the multiindex notation. Let $\mathbb{Z}_+$ denote the set of nonnegative integers, and let small Greek letters denote the multiindex vectors i.e. vectors with nonnegative integer components. For any multiindices

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{Z}_+^m, \quad \beta = (\beta_1, \beta_2, \cdots, \beta_m) \in \mathbb{Z}_+^m,$$

and a vector $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$, let

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_m, \quad \alpha! := \alpha_1! \alpha_2! \cdots \alpha_m!\ , \quad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m},$$

$$\binom{n}{\alpha} := \begin{cases} \frac{n!}{\alpha!(n-|\alpha|)!}, & 0 \leq \alpha, |\alpha| \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\alpha \leq \beta$ denotes the relation $\leq$ componentwise i.e. $\alpha_i \leq \beta_i$, all i, and further let $\alpha < \beta$ be $\alpha \leq \beta$ with at least one $\alpha_i < \beta_i$. The generalised binomial coefficient is given by

$$\binom{\alpha}{\beta} := \begin{cases} \prod_{j=1}^{m} \binom{\alpha_j}{\beta_j}, & 0 \leq \beta \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$