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MONOTONE APPROXIMATION TO A SYSTEM WITHOUT MONOTONE NONLINEARITY*

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Abstract

A monotone approximation is proposed for a system without monotone nonlinearity. A new concept of ordered pair of supersolution and subsolution is introduced, and then the existence of numerical solutions is studied. A new monotone iteration is provided for solving the resulting problem. An approximation with high accuracy is investigated. The corresponding iteration possesses geometric convergence rate. The numerical results support the theoretical analysis.

Key words: Monotone approximation, Systems without monotone nonlinearity.

1. Introduction

Due to the development of various studies in electromagnetism, biology and some other fields, nonlinear systems have been paid extensive attention both analytically and numerically, e.g., see [1–12]. As we know, a reasonable numerical method should not only have the approximation error of higher order, but also preserve the main feature of the original problem. In this case, the numerical results might fit the physical process better. Since the usual approximations simulate the maximum principle, they are of positive-type. Thus they possess only the second order. In [13], the authors proposed a new approach for a nonlinear equation. This approach simulates the comparison principle and thus provides the higher accuracy. Later, the authors generalized this approach to nonlinear systems, e.g., see [14]. However, the corresponding analysis is valid only when the nonlinear terms are monotone in some sense. This fact limits the application of this new approach. So, the question whether it is possible to develop this approach for some systems without any monotone nonlinearity, is natural and interesting. In this paper we investigate this problem. The answer is positive.

The outline of this paper is as follows. In Section 2, we present the monotone approximation for a system without any monotone nonlinearity. Then we introduce a new concept of ordered pair of supersolution and subsolution for the resulting problem, and study the existence of numerical solutions. Finally, we propose a new iteration for solving the resulting problem. In Section 3, we investigate a monotone approximation on uniform mesh. Especially, we give a sufficient condition ensuring the convergence

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of iteration to the unique solution of the corresponding problem in some cone. We also show that the iteration has the geometric convergence rate under some reasonable conditions. In the final section, we list the numerical results which coincide with the theoretical analysis in the previous sections, and show the advantages of this method.

2. General Framework

Let $I = \{x|\ 0 < x < 1\}$ and $\overline{I}$ be its closure. $u = (u_1, u_2, \cdots, u_n)^T$ denotes a vector function of x. The given function $f(x,u) \in [C^0(\overline{I} \times \mathbf{R}^n) \cap C^1(I \times \mathbf{R}^n)]^n$ has the components $f_i(x,u)$. Also let $a_i(x) \in C^1(I)$ and assume that for certain positive constants $\alpha_0 \le \alpha_1$, $\alpha_0 \le a_i(x) \le \alpha_1$ for $x \in I$ and $1 \le i \le n$. Moreover suppose that $\left|\frac{\mathrm{d}a_i}{\mathrm{d}x}(x)\right|$ is bounded for $x \in I$ and $1 \le i \le n$. Furthermore, let $l = \mathrm{diag}(l_1, \cdots, l_n)$ with

$$l_i u_i(x) = -(a_i(x)u_i'(x))', \quad u_i'(x) = \frac{\mathrm{d}u_i}{\mathrm{d}x}(x), \qquad 1 \le i \le n.$$

Set $F_{i,j}(x,u) = \frac{\partial f_i}{\partial u_j}(x,u)$, $1 \le i,j \le n$. We consider the following coupled problem, i.e., finding $u(x) \in [C^0(\overline{I}) \cap C^2(I)]^n$ such that

$$\begin{cases} lu(x) + f(x,u(x)) = 0, & x \in I, \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Such a problem arises in many fields, e.g., see [15]. It is well known that if $u(x) \in [C^0(\overline{I}) \cap C^2(I)]^n$ and

$$\begin{cases} lu(x) \ge 0, & x \in I, \\ u(0) \ge 0,\ u(1) \ge 0, \end{cases}$$

then $u(x) \ge 0$ for $x \in \overline{I}$. Conversely, if the reversed inequalities hold in the above problem, then $u(x) \le 0$ for $x \in \overline{I}$.

It is not difficult to prove the existence of solutions of (2.1) under some conditions. To solve (2.1) numerically, we introduce a set of mesh points $\{x_p\}_0^N$ such that

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1.$$

For each p, let $I_p = (x_{p-1}, x_p)$, $\overline{I}_p = [x_{p-1}, x_p]$, $h_p = x_p - x_{p-1}$, and $h = \max_{1 \le p \le N} h_p$. We also set $I_h = \{x_p\}_1^{N-1}$ and $\overline{I}_h = \{x_p\}_0^N$. Next let $l_h = \mathrm{diag}(l_{h,1}, \cdots, l_{h.n})$ and $P_h = \mathrm{diag}(P_{h,1}, \cdots, P_{h.n})$ be certain linear discrete operators. Then the corresponding discrete problem might be stated as follows, i.e., finding $u_h(x) = (u_{h.1}(x), \cdots, u_{h,n}(x))^T$ such that

$$\begin{cases} l_h u_h(x) + P_h f(x, u_h(x)) = 0, & x \in I_h, \\ u_h(0) = u_h(1) = 0. \end{cases} \tag{2.2}$$

We say that (2.2) is a monotone approximation, if the following conditions are fulfilled,