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A FAMILY OF HIGH-ODER PARALLEL ROOTFINDERS FOR POLYNOMIALS[*1)]

Shi-ming Zheng
(*Department of Mathematics, Xixi Campus, Zhejiang University, Hangzhou, 310028, China*)

Abstract

In this paper we present a family of parallel iterations of order $m+2$ with parameter $m=0,1\cdots$ for simultaneous finding all zeros of a polynomial without evaluation of derivatives, which includes the well known Weierstrass-Durand-Dochev-Kerner and Börsch-Supan-Nourein iterations as the special cases for $m=0$ and $m=1$, respectively. Some numerical examples are given.

Key words: Parallel iteration, zeros of polynomial, order of convergence

1. Introduction

Let

$$f(t)=\sum_{i=0}^{n} a_i t^{n-i}=\prod_{j=1}^{n}(t-\xi_i), \quad a_0=1 \tag{1}$$

be a monic complex polynomial of degree n with zeros $\xi_1,\cdots,\xi_n$. Some authors have studied the parallel iterations without evaluation of derivatives for simultaneous finding all zeros of $f(t)$ (see [1]-[10]). The famous one is Weierstrass-Durand-Dochev-Kerner iteration

$$x_i^{k+1}=x_i^k-u_i^k \quad i=1,2,\cdots,n, \quad k=0,1,\cdots, \tag{2}$$

where x_i^k is the k-th approximation of $\xi_i (1\le i\le n)$ and

$$u_i^k=\frac{f(x_i^k)}{\prod\limits_{j\neq i}(x_i^k-x_j^k)}, \quad i=1,\cdots,n, \quad k=0,1,\cdots, \tag{3}$$

which does not require any information of derivatives and was presented independently by Weierstrass[7], Durand[2], Dochev[3] and Kerner[4]. It is well known that the convergence of (2) is quadratic if $\xi_i\neq\xi_j$ for $i\neq j$. Another one is

$$x_i^{k+1}=x_i^k-\frac{u_i^k}{1+\sum\limits_{j\neq i}\dfrac{u_j^k}{x_i^k-x_j^k}}, \quad i=1,2,\cdots,n, \quad k=0,1,\cdots, \tag{4}$$

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which was derived by Börsch-Supan[1], later, by Nourein[5], and the convergence is cubic if $\xi_i \neq \xi_j$ for $i \neq j$.

In this paper we present a family of parallel iterations of order $m+2$ with parameter $m = 0, 1 \cdots$, which includes Weierstrass-Durand-Dochev-Kerner iteration (2) and Börsch-Supan-Nourein iteration (4) as the special cases for $m = 0$ and $m = 1$, respectively. Some numerical examples are given in section 4.

2. Construction of the Iterations

For purposes of brevity, all formulas, sums and products (such as in (2), (3) and (4) above) involving indices i, j and ν will assume the range $1, 2, \cdots, n$ and the iterative index $k = 0, 1, \cdots$, unless explicit stated otherwise. Naturally, we always regard $\sum_{l=\nu}^{\mu}(\cdots) = 0$ for $\mu < \nu$. Moreover, we simply write $x_i, u_i, \cdots$ for $x_i^k, u_i^k, \cdots$ and x_i^+ for x_i^{k+1}.

To construct the family of the iterations we first give the following

Proposition. *Let $x_1, x_2, \cdots, x_n \notin \{\xi_1, \xi_2, \cdots, \xi_n\}$ be distinct. Define*

$$u_j = \frac{f(x_j)}{\prod_{\nu \neq j}(x_j - x_\nu)}. \tag{5}$$

$$\begin{cases} \delta_i = & x_i - \xi_i \\ S_{il} = & \sum_{j \neq i} \frac{u_j}{(x_i - x_j)^l}, \ l = 1, 2, \cdots, \\ T_{im} = & \sum_{l=1}^{m} S_{il} \delta_i^{l-1}, \ m = 0, 1, \cdots, \\ R_{im} = & \delta_i^m \sum_{j \neq i} \frac{u_j}{(x_i - x_j)^m (\xi_i - x_j)}, \ m = 0, 1, \cdots. \end{cases} \tag{6}$$

Then for all $m = 0, 1, \cdots$ the fixed point relation

$$\xi_i = x_i - \delta_i = x_i - \frac{u_i}{1 + T_{im} + R_{im}}, \ m = 0, 1, \cdots \tag{7}$$

holds.

proof. Using Lagrange interpolation, we have

$$f(t) = \left(\sum \frac{u_j}{t - x_j} + 1\right) \prod (t - x_j). \tag{8}$$

Substituting $t = \xi_i \notin \{x_1, \cdots, x_n\}$ into (8) and observing $f(\xi_i) = 0$, we obtain

$$\frac{u_i}{\xi_i - x_i} + 1 + \sum_{j \neq i} \frac{u_j}{\xi_i - x_j} = 0, \tag{9}$$

$$\delta_i = x_i - \xi_i = \frac{u_i}{1 + \sum_{j \neq i} \frac{u_j}{\xi_i - x_j}}. \tag{10}$$