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BROYDEN'S METHOD FOR SOLVING VARIATIONAL INEQUALITIES WITH GLOBAL AND SUPERLINEAR CONVERGENCE *

Yu-fei Yang   Dong-hui Li
(*Department of Applied Mathematics, Hunan University, Changsha 410082, China*)

Abstract

In this paper, we establish a quasi-Newton method for solving the KKT system arising from variational inequalities. The subproblems of the proposed method are lower-dimensional mixed linear complementarity problems. A suitable line search is introduced. We show that under suitable conditions, the proposed method converges globally and superlinearly.

Key words: Variational inequality, quasi-Newton method, global convergence, superlinear convergence

1. Introduction

We are concerned with the following variational inequality problem of finding an $x \in X$ such that

$$(y-x)^T f(x) \geq 0, \quad \forall y \in X, \tag{1.1}$$

where $f: R^n \to R^n$ is assumed to be a continuously differentiable function, and $X \subseteq R^n$ is specified by

$$X = \{x \in R^n : g_i(x) \leq 0, i = 1, 2, \ldots, m; \;\; h_j(x) = 0, j = 1, 2, \ldots, l\}, \tag{2.2}$$

where $g_i : R^n \to R$ and $h_j : R^n \to R$ are twice continuously differentiable functions. The variational inequality (1.1) is denoted by $\mathrm{VI}(X, f)$. An important special case of $\mathrm{VI}(X, f)$ is the so-called nonlinear complementarity problem ($\mathrm{NCP}(f)$) with $X = R^n_+ = \{x \in R^n \,|\, x \geq 0\}$. Variational inequality and nonlinear complementarity problems have attracted many people's attention because of their wide practice background. We refer to [5] for a review on it.

Originated by Josephy [7, 8], quasi-Newton methods are now an important class of iterative methods for solving $\mathrm{VI}(X, f)$ and $\mathrm{NCP}(f)$, and have attracted of many veseachers (see [2, 5, 6, 7, 8, 10, 13, 16, 18] etc.).

One kind of quasi-Newton methods is the linearized quasi-Newton methods where the $\mathrm{VI}(X, f)$ and $\mathrm{NCP}(f)$ are approximated successively by a sequence of linear problems $\mathrm{VI}(X, f^k)$ and $\mathrm{NCP}(f^k)$ (e.g., see [8] and [13]), where

$$f^k(x) = f(x_k) + B_k(x - x_k),$$

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and B_k is an $n \times n$ matrix. Another kind of quasi-Newton methods is established based on equivalent nonsmooth equations of VI(X, f) and NCP(f) (e.g., see [2, 6, 16]). The local convergence properties of these quasi-Newton methods are very similar to those of quasi-Newton methods for smooth equations and optimizations. In particular, they converge superlinearly. However, so far the studies for quasi-Newton methods focused on their local behavior. This paper devotes to develop a globally convergent quasi-Newton method. To do this, we simply review the related globally convergent Newton methods for solving VI(X, f).

The globally convergent Newton methods have developed vary fast in the past two decades. Using appropriate reformulations, there have been established various kinds of damped Newton methods (e.g., see [11, 12, 15, 17]). The ideas of Newton methods for solving smooth equations and mathematical programming were applied to solve variational inequalities successfully. Basically, in these methods, Armijo-type line search was used. Under suitable conditions, these methods converge globally and superlinearly/quadratically. For quasi-Newton methods, however, Armijo-type line search seems inappropriate because its implementation relies on the calculation of $\nabla f(x)$. Therefore, to establish globally convergent quasi-Newton methods, it seems necessary to adopt new line search technique. In [18], on the basis of the derivative-free line search given by Griewank [4] on quasi-Newton methods for smooth equations, a derivative-free line search was proposed, and a quasi-Newton method for solving NCP(f) was proposed. Under certain conditions, the method converges globally and superlinearly. This line search was further studied in [10]. In this paper, we adopt the same line search to establish a Broyden-like method for solving VI(X, f). Under suitable conditions, we prove its global and superlinear convergence. The method in this paper is quite different from the ones in [10] and [18]. The advantages of method in this paper are: first, the method is suitable to solving general variational inequality problems. Second, the subproblem is a lower-dimensional mixed linear complementarity problem. Moreover, as we will show in section 4, the method converges superlinearly without strict complementarity assumption.

The organization of the paper is as follows. In the next section, we do some preliminaries, and describe the method. In section 3 and 4, we prove global and superlinear convergence of the proposed method.

2. Preliminaries

In this section, we do some preliminaries. We first introduce the Karush-Kuhn-Tucker system for VI(X, f). It is well-known (e.g., see [5] or [12]) that if x is a solution of $VI(X, f)$ and a suitable constraint qualification is satisfied at x, then there exist multiplier vectors $u \in R^m$ and $v \in R^l$ such that the following mixed complementarity conditions are satisfied:

$$\begin{cases} f(x) + \nabla g(x)u + \nabla h(x)v = 0, \\ u \geq 0, \quad g(x) \leq 0 \quad \text{and} \quad u^T g(x) = 0, \\ h(x) = 0, \end{cases} \tag{2.1}$$