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BLOCKWISE PERTURBATION THEORY FOR 2×2 BLOCK MARKOV CHAINS[*1)]

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Abstract

Let P be a transition matrix of a Markov chain and be of the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

The stationary distribution π^T is partitioned conformally in the form (π_1^T, π_2^T). This paper establish the relative error bound in π_i^T $(i = 1, 2)$ when each block P_{ij} get a small relative perturbation.

Key words: Blockwise perturbation, Markov chains, stationary distribution, error bound

1. Introduction

The sensitivity of the stationary distribution to general perturbations in a transition matrix have been addressed by many authors [1], [2], [4], [6]. Let P amd $\tilde{P} = P + F$ be irreducible transition matrices with respective stationary distributions π^T and $\tilde{\pi}^T$ satisfying

$$P\mathbf{1} = \tilde{P}\mathbf{1} = \mathbf{1},\ \pi^T P = \pi^T,\ \tilde{\pi}^T P = \tilde{\pi}^T,\ \pi^T \cdot \mathbf{1} = \tilde{\pi}^T \cdot \mathbf{1} = 1.$$

Here, we denote by $\mathbf{1}$ (a bold one) the vector of all ones. In later discussion, we give its size with a subscript (e.g. $\mathbf{1})_n$ for the vector with n entries) explicitly when necessary. It is well known that

$$\|\pi - \tilde{\pi}\| \leq \|F\| \cdot \|A^{\#}\|, \tag{1}$$

where $A^{\#}$ is the group inverse of $A = I - P$, and $\| * \|$ denotes the infinity norm.

For some Markov chains, such as nearly uncoupled Markov chains, $\|A^{\#}\|$ is very large, which means that small perturbations in P can cause severe perturbations in π. However, the stationary distribution π can be insensitive to some special perturbation F. See [5], [9].

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The purpose of this paper is to analyze the effects of small blockwise relative perturbation to a 2×2 block transition matrix. More precisely, let P has the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

where P_{11} and P_{22} are square matrices of order n_1 and n_2, and let F, π^T and $\tilde{\pi}^T$ be partitioned conformally as

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad \pi^T = (\pi_1^T, \pi_2^T) \quad \text{and} \quad \tilde{\pi}^T = (\tilde{\pi}_1^T, \tilde{\pi}_2^T).$$

Under the condition that P_{ii} $(i = 1, 2)$ is irreducible and

$$\|F_{ij}\| \leq \eta \cdot \|P_{ij}\|, \quad i, j = 1, 2,$$

we are to bound $\|\pi_i^T - \tilde{\pi}_i^T\| / \|\pi_i^T\|$. Under certain condition, we will show that this relative error can be small even when $\|A^{\#}\|$ is large, or when $\|\pi_1\|$ is far more large (or less) than $\|\pi_2\|$. In [8], G. W. Stewart provided the relative error bound for π_2 when P is the transition matrix of a nearly transient Markov chain, i.e., $\|P_{12}\|$ is very small and $\|P_{21}\|$ is of magnitude one. In his analysis, he assumed that $F_{12} = 0$ and $\|\pi_1 - \tilde{\pi}_1\| / \|\pi_1\|$ is small. In this paper, these restrictions are deleted. The only assumption is that P_{11} and P_{22} are irreducible.

2. Some Basic Lemmas

In this section, we present some basic lemmas for Markov chains. These lemmas are important tools in deriving the main result of this paper.

Lemma 1. *Let x and y are n-vectors satisfy $y^T x = 1$. Then there are matrices J and K such that*

$$(x, J)^{-1} = \begin{pmatrix} y^T \\ K^T \end{pmatrix}.$$

Moreover, $\|J\|_2 = 1$ and $\|K\|_2 = \|x\|_2 \cdot \|y\|_2$.

Proof. See [7].

From the relation between ∞-norm and 2-norm,

$$\frac{1}{\sqrt{n}} \|B\|_\infty \leq \|B\|_2 \leq \sqrt{m} \|B\|_\infty, \quad B \in \mathbb{C}^{m \times n},$$

we have

$$\|J\|_\infty \leq \sqrt{n-1} \|J\|_2 = \sqrt{n-1} \tag{2}$$

and

$$\|K\|_\infty \leq \sqrt{n-1} \|K\|_2 = \sqrt{n-1} \cdot \|x\|_2 \cdot \|y\|_\infty. \tag{3}$$

The following two lemmas can be found in [8]