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LEAST-SQUARES MIXED FINITE ELEMENT METHOD FOR SADDLE-POINT PROBLEM*1)

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Abstract

In this paper, a least-squares mixed finite element method for the solution of the primal saddle-point problem is developed. It is proved that the approximate problem is consistent ellipticity in the conforming finite element spaces with only the discrete BB-condition needed for a smaller auxiliary problem. The abstract error estimate is derived.

Key words: least-squares method, mixed finite element approximation, saddle-point problem

1. Introduction

There are many work to investigate the stability of the mixed finite element method for the saddle-point problems, i.e., to construct the finite element spaces, such that the so-called discrete BB-codition is satisfied (c.f. [1],[2],[7],[8] and the references therein). To circumvent the discrete BB-conditon, recently there has been an increased interest in use of least-squares approach for the solution of the mixed finite element approximation of the saddel-point problem (c.f.[3]–[6],[10],[12] and [13]).In this aspect, the saddle-point problem (such as the Stokes problem) is reduced, in general, the first order system by introducing auxiliary variables (such as the stress for the Stokes problem). Thus the bilinear form, in the least-squares mixed finite element approximation for the saddel-point problem, is coercive in the conforming finite element spaces, and the discrete BB-condition is not required.

In this paper, the least-squares mixed approach, a least-squares residual minimization, is introduced for the primal saddel-point problem directly, without use of any auxiliary variables. The ellipticity in the finite element spaces for the least-squares mixed finite element approximation of the primal saddel-point problem is guaranteed, under the assumption of the discrete BB-condition being satisfied for a smaller auxiliary problem, instead for the primal saddel-point problem. And under the same assumption presented previously, the abstract error estimate is derived.

The paper is organized as follows. In section 2, we formulate the least-squares mixed problem and proved the coerciveness of the bilinear form in the case of the BB-condition satisfied for the primal saddle-point problem. In section 3, as a bridge for theoretical

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analysis but not for practical computing, a semi-finite element approximation is presented. Without any discrete BB-condition being required, we derive the stability and the abstract error estimate, which will be used in the next section. In section 4 and 5, the real finite element approximation is presented, the stability and the abstract error estimate are derived with only the discrete BB-conditon being required for a smaller auxiliary problem.

2. The Least-Squares Mixed Formulation for The Saddel-Point Problem

Let V, Q be two real Hilbert spaces with norm $\|\cdot\|_V$ and $\|\cdot\|_Q$ respectively, the norm $\|(\cdot,\cdot)\|_{V\times Q} = (\|\cdot\|_V^2 + \|\cdot\|_Q^2)^{\frac{1}{2}}$, V' and Q' denote the dual spaces of V and Q with norm $\|\cdot\|_{V'}$ and $\|\cdot\|_{Q'}$ respectively, and the dualities of V' and V, Q' and Q be denoted by $<\cdot,\cdot>$. For any given $f \in V', \chi \in Q'$, we consider the following mixed variational equations of the saddel-point problem

$$\begin{cases} \text{to find } \ (u,p) \in V \times Q, & \text{such that} \\ a(u,v) + b(v,p) & =< f, v > \quad \forall \ v \in V, \\ b(u,q) & =< \chi, q > \quad \forall \ q \in Q, \end{cases} \tag{2.1}$$

where $a(\cdot,\cdot) : V \times V \to R$ is a symmetric, continuous bilinear form, and V-elliptic, e.i., there exists $\alpha =$ const. > 0, such that

$$a(v,v) \geq \alpha \|v\|_V^2 \qquad \forall \ v \in V, \tag{2.2}$$

$b(\cdot,\cdot) : V \times Q \to R$ is a continuous bilinear form with the following BB-condition: there exists $\beta =$ const. > 0, such that

$$\sup_{v \in V} \frac{b(v,q)}{\|v\|_V} \geq \beta \|q\|_Q \qquad \forall \ q \in Q. \tag{2.3}$$

Then the following theorem is well known (c.f.[7], [8])

Theorem 2.1. *Assume that* (i) *V, Q are the real Hilbert spaces,* (ii) *the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous with V-ellipticity* (2.2), *and the bilinear form $b(\cdot,\cdot)$ is continuous with BB-condition* (2.3). *Then the abstract problem* (2.1) *has one and only one solution (u,p), and the following inequality holds*

$$\|u\|_V^2 + \|p\|_Q^2 \leq C(\|f\|_{V'}^2 + \|\chi\|_{Q'}^2), \tag{2.4}$$

where $C =$ Const.> 0.

Since the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and V-elliptic, the space V equipped the inner product $a(\cdot,\cdot)$, also denoted $(\cdot,\cdot) = a(\cdot,\cdot)$, is a Hilbert space, and the corresponding norm is also denoted by $\|\cdot\|_V$. The dual space of V, equipped the inner product $(\cdot,\cdot) = a(\cdot,\cdot)$, is also denoted by V'. It is easily seen that the inequalities (2.2)–(2.4) hold.

To formulate the least-squares mixed form of the problem (2.1), it is needed to introduce the following operators (c.f.[7],[8])

$$\begin{cases} A \in \mathcal{L}(V;V') : \qquad a(u,v) =< Au, v >, \\ B \in \mathcal{L}(V;Q'), \ \ B^T \in \mathcal{L}(Q;V') : \qquad b(v,q) =< Bv, q >=< v, B^T q > . \end{cases} \tag{2.5}$$