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ON TRIANGULAR C^1 SCHEMES: A NOVEL CONSTRUCTION*[1)]

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Abstract

In this paper we present a C^1 interpolation scheme on a triangle. The interpolant assumes given values and one order derivatives at the vertices of the triangle. It is made up of partial interpolants blended with corresponding weight functions. Any partial interpolant is a piecewise cubics defined on a split of the triangle, while the weight function is just the respective barycentric coordinate. Hence the interpolant can be regarded as a piecewise quartic. We device a simple algorithm for the evaluation of the interpolant. It's easy to represent the interpolant with B-net method. We also depict the Franke's function and its interpolant, the illustration of which shows good visual effect of the scheme.

Key words: Spline, interpolation scheme, partial interpolants, barycentric coordinates, splits, B-net

1. Introduction

The smooth interpolation on a triangulation of a planar region is of great importance in most applied areas, such as computation of finite element method, computer aided (geometric) design and scattered data processing.

Let Δ be a triangulation of a polygonal domain $\Omega \subset R^2$ and Δ_0, Δ_1 and Δ_2 the sets of vertices, edges and triangles in Δ respectively. Usually the triangulation in practice is formed by a mass of scattered nodes that, covered by the region Ω, are carrying similar types of data, i.e., positions and derivatives.

We have interest in this paper only the C^1 case and consider the following interpolation problem

$$\begin{cases} \text{For } f \in C^1(\Omega), \text{ find a function } g \in C^1 \text{ such that} \\ g(v) = f(v), \\ \frac{\partial}{\partial x} g(v) = \frac{\partial}{\partial x} f(v), \quad \frac{\partial}{\partial y} g(v) = \frac{\partial}{\partial y} f(v), \\ \text{for } v \in \Delta_0. \end{cases} \tag{1}$$

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To solve problem (1), one efficient way is the *local* approach(cf. [3]). It is based on a single triangle. Once a solution of the problem restricted on the triangle is found, and it satisfies the property that the interpolants on any two adjacent triangles connect with C^1 smoothness, the triangle and its interpolant are representative and they together form an interpolation model or a C^1 interpolation scheme. Hence on Ω, a C^1 interpolant is constructed by piecing together the interpolants on all the triangles of triangulation Δ. The interpolation scheme such constructed has an another advantage: any change on a vertex affects only the interpolants on the star region of the vertex. Therefore it shows local significance.

To construct the interpolants, functions that are easy to formulate and evaluate are preferable, such as polynomials, rationals, and their piecewise versions, i.e. splines(see [4] and [6]).

Therefore we consider only the following problem for triangle $T = A_1A_2A_3$. Let e_i denote the opposite edge of A_i and n_i the outer normal vector of e_i, $i = 1, 2, 3$.

$$\begin{cases} \text{For } f \in C^1(T), \text{ find a function } g \in C^1(T) \text{ such that} \\ g(A_i) = f(A_i), \\ \frac{\partial}{\partial x} g(A_i) = \frac{\partial}{\partial x} f(A_i), \quad \frac{\partial}{\partial y} g(A_i) = \frac{\partial}{\partial y} f(A_i), \\ \text{and } g \text{ and } \frac{\partial}{\partial n_i} g \text{ on } e_i \text{ are univariate polynomials} \\ \text{of degrees 3 and 2 respectively, for } i = 1, 2, 3. \end{cases} \tag{2}$$

If the interpolant is a polynomial, derivatives at the vertices of the triangle of order 2 are needed and hence beyond the conditions given in (2)(cf. [7] and [8]). It also introduces a drawback that the degree of the polynomial reaches as high as 5(see [7] and [8]).

By splitting the triangle in HCT type(see [5]), one can find a solution of (2). The interpolant is a piecewise cubics. For a C^1 scheme, we have shown that the triangle must be subdivided and each angle of the triangle is split into 2 parts[8]. This is a crucial rule.

The splitting method lowers the order of interpolation data needed and the degree of interpolant polynomial. But it increases the computational complication. If we want the triangle is less split, the rationals instead of polynomials are suitable candidates. But rationals often introduce singular points which will cause unstable in evaluation.

So, some *compromise* between splits and rationals or polynomials will better the cases.

In this paper we find a solution of (2) by a hybrid of polynomials and splits. The interpolant of (2) is made up of partial interpolants blended with corresponding weight functions. Any partial interpolant is a piecewise cubics defined on a split of the triangle, while the weight function is just the respective barycentric coordinate. Hence the interpolant can be regarded as a piecewise quartics. We device a simple algorithm for the evaluation of the interpolant. It's easy to represent the interpolant with B-net