Journal of Computational Mathematics, Vol.18, No.5, 2000, 551–560.

CASCADIC MULTIGRID FOR PARABOLIC PROBLEMS[*1)]

Zhong-ci Shi   Xue-jun Xu
(*State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, we develop the cascadic multigrid method for parabolic problems. The optimal convergence accuracy and computation complexity are obtained.

Key words: Cascadic multigrid, Finite element, Parabolic problem.

1. Introduction

Bornemann and Deuflhard [2][3] have presented a new type of multigrid methods, the so-called cascadic multigrid. Compared with usual multigrid methods, it requires no coarse grid corrections at all that may be viewed as a "one way" multigrid. Another distinctive feature is performing more iterations on coarser levels so as to obtain less iterations on finer levels. Numerical experiments show that this method is very effective for second order elliptic problems.

In the paper, we will consider the cascadic multigrid for parabolic problems. Here we must treat the effect of the time discrete step. As pointed out in [1], for a small time step $\tau \le O(h^2)$, where h is the space mesh size, some standard iterative methods, like the Richardson iteration can guarantee a good convergence for the discrete system. But for a relative large time step τ, [1] recommended multigrid methods, see [4] for details. Now we consider to use the cascadic multigrid. Similar to the second order elliptic problem, it is proved that the cascadic multigrid with CG iteration as a smoother is accurate with the optimal complexity in 3D and 2D, and nearly optimal in 1D. As for other traditional iterative methods, like the Richardson iteration, the cascadic multigrid still yields the optimal accuracy and complexity in 3D, 2D and in a certain case of 1D. Notice that for the second order elliptic problem, the cascadic multigrid with these iterative methods gives the optimal accuracy and computation complexity only in 3D and nearly optimal in 2D. They cannot be used for 1D.

[*] Received November 12, 1998.
[1)] Subsidized by the Special Funds for Major State Basic Research Projects.

2. Model Problem and Its Finite Elment Approximation

Consider the following parabolic problem: to find $u(x,t)$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u & = f \quad in \quad \Omega \times [0,T], \\ u(x,t) & = 0 \quad in \quad \partial\Omega \times [0,T], \\ u(x,0) & = u_0(x), \end{cases} \tag{2.1}$$

where $\Omega \subset R^d$ $(d = 1,2,3)$ is a bounded domain, $f \in L^2(\Omega)$. $\mathcal{L}$ is an elliptic operator

$$\mathcal{L}u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}). \tag{2.2}$$

Here $a_{ij}(x)$ satisfies

$$c\xi^t\xi \leq \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \leq C\xi^t\xi \quad \forall x \in \Omega, \xi \in R^d, \tag{2.3}$$

where c,C are positive constants.

The variational form of (2.1) is to find $u \in H_0^1(\Omega)$, $u(x,0) = u_0(x)$ such that

$$(\frac{\partial u}{\partial t}, v) + B(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \quad t \in [0,T], \tag{2.4}$$

where the bilinear form B is

$$B(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} dx \quad \forall u,v \in H^1(\Omega)$$

and

$$(f,v) = \int_\Omega fv dx.$$

We refer the notations of Sobolev space to [5] for details. It is easily seen that the bilinear form $B(u,v)$ is

(1). bounded, i.e. $|B(u,v)| \leq C|u|_1|v|_1 \quad \forall u,v \in H_0^1(\Omega)$.

(2). elliptic, i.e. $|B(u,u)| \geq C|u|_1^2 \quad \forall u \in H_0^1(\Omega)$.

We use the backward Euler scheme and Crank-Nicolson scheme for the time discretization [8]. Both schemes are absolutely stable [6]. Let $\triangle t_n$ be the n^{th} time step and M the number of steps, then $\sum_{n=1}^{M} \triangle t_n = T$. We lead to the following problem: for a given function $g_{n-1} \in H^{-1}(\Omega)$, find $w \in H_0^1(\Omega)$ such that

$$A_\tau(w,v) = \tau^{-1}(w,v) + B(w,v) = (g_{n-1},v) \quad \forall v \in H_0^1(\Omega), \tag{2.5}$$