Journal of Computational Mathematics, Vol.19, No.1, 2001, 101–112.

COMPOSITE LEGENDRE–LAGUERRE APPROXIMATION IN UNBOUNDED DOMAINS*

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

Composite Legendre–Laguerre approximation in unbounded domains is developed. Some approximation results are obtained. As an example, a composite spectral scheme is provided for the Burgers equation on the half line. The stability and convergence of proposed scheme are proved strictly. Two-dimensional exterior problems are discussed.

Key words: Composite spectral approximation, Unbounded domains, Exterior problems.

1. Introduction

Many problems in science and engineering are set in unbounded domains. There are several ways for their numerical simulations. We may restrict calculations to some bounded domains with certain artificial boundary conditions. But they induce errors. In particular, they affect the wave propagations in revolutionary problems. In opposite, if we use spectral methods associated with orthogonal systems of polynomials in unbounded domains, then we could avoid this trouble, e.g., see Maday, Pernaud-Thomas and Vandeven [1], Funaro [2], Funaro and Kavian [3], Boyd [4], Guo [5], and Guo and Shen [6]. Funaro and Kavian [3] proved the convergence for some linear problems, by using the Hermite functions. Recently Guo [5] proved the stability and the convergence of spectral approximations to nonlinear problems, using the Hermite polynomials. While Guo and Shen [6] analyzed the errors of the Laguerre spectral schemes for several nonlinear problems. But there are still some remaining problems. Firstly, in order to get the same accuracy, the Hermite and Laguerre methods need more regularities of solutions of differential equations, than the Legendre and Chebyshev methods for the same problems in the corresponding bounded subdomains. However, the solutions may change rapidly in certain bounded subdomains. For instance, the solutions might be less smooth near some corners. On the other hand, most of multiple-dimensional problems are set in non-rectangular domains, and so the standard Hermite and Laguerre approximations are not available for them. In particular, for exterior problems, the domains are never rectangular, and the solutions change very rapidly near the obstacles usually. One of reasonable ways for resolving such problems is to use spectral domain decomposition method, e.g., see Quarteroni[7], and Coulaud, Funaro and Kavian [8]. For instance, we may divide the domain into several subdomains, and then use the Legendre approximations in the bounded subdomains, and use the Laguerre approximations

* Received November 11, 1998.

in the remaining parts. But so far, there is no theoretical results in this field. The aim of this paper is to investigate a new spectral domain decomposition method, called as composite spectral method. For simplicity of analysis, we first consider a one-dimensional model in detail, and then discuss two-dimensional problems in non-rectangular domains, and exterior problems. In the next section, we divide the half line to a finite subinterval and a infinite subinterval, and then construct a composite Legendre–Laguerre spectral scheme for the Burgers equation on the half line. We prove some composite imbedding inequalities and approximation results in Section 3, which play important roles in analysis of the composite spectral method. In section 4, we use the results in the previous section to prove the stability and the convergence of proposed scheme strictly. The final section is for two-dimensional problems. We consider a non-rectangular unbounded domain and an exterior problem.

2. The Composite Spectral Scheme

Let $I = (a, b), -\infty \le a < b \le \infty$, and $\chi(x)$ be certain weight function in the usual sense. For any $1 \le p \le \infty$, define

$$L^p_\chi(I) = \{v \mid \|v\|_{L^p_{\chi,I}} < \infty\}$$

where

$$\|v\|_{L^p_{\chi,I}} = \begin{cases} (\int_I |v(x)|^p \chi(x) dx)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in I} |v(x)|, & \text{if } p = \infty. \end{cases}$$

In particular, we denote by $(u, v)_{\chi,I}$ and $||v||_{\chi,I}$ the inner product and the norm of the space $L^2_\chi(I)$. Further let $\partial_x v(x) = \frac{\partial}{\partial_x} v(x), etc.$. For any non-negative integer m,

$$H^m_\chi(I) = \{v \mid \partial^k_x v \in L^2_\chi(I),\ 0 \le k \le m\}$$

equipped with the following semi-norm and norm

$$|v|_{m,\chi,I} = \|\partial^m_x v\|_{\chi,I}, \quad \|v\|_{m,\chi,I} = \left(\sum_{k=0}^m |v|^2_{k,\chi,I}\right)^{\frac{1}{2}}.$$

For any $r > 0$, we define the space $H^r_\chi(I)$ with the norm $\|v\|_{r,\chi,I}$ by the space interpolation as in Adams [9]. Moreover let

$$H^1_{0,\chi}(I) = \{v \mid v \in H^1_\chi(I) \text{ and } v(a) = \lim_{x\to b} \chi^{\frac{1}{2}}(x)v(x) = \lim_{x\to b} \chi^{\frac{1}{2}}(x)\partial_x v(x) = 0\ \}.$$

For $\chi(x) \equiv 1$, we denote $H^r_\chi(I), H^r_{0,\chi}(I), |v|_{r,\chi,I}, \|v\|_{r,\chi,I}, (u, v)_{\chi,I}$ and $||v||_{\chi,I}$ by $H^r(I), H^r_0(I), |v|_{r,I}$, $\|v\|_{r,I}, (u, v)_I$ and $||v||_I$, respectively. In addition, $||v||_{\infty,I}$ stands for $||v||_{L^\infty(I)}$.

Let $\Lambda = (-1, \infty)$, and consider the Burgers equation on the half line as follows

$$\begin{cases} \partial_t U(x,t) + \frac{1}{2}\partial_x(U^2(x,t)) - \mu\partial^2_x U(x,t) = f(x,t), & x \in \Lambda, 0 < t \le T, \\ U(-1,t) = d(t), & 0 < t \le T, \\ \lim_{x\to\infty} U(x,t) = \lim_{x\to\infty} \partial_x U(x,t) = 0, & 0 < t \le T, \\ U(x,0) = U_0(x), & x \in \Lambda \end{cases} \tag{2.1}$$