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NON-STATIONARY STOKES FLOWS UNDER LEAK BOUNDARY CONDITIONS OF FRICTION TYPE*

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

This paper is concerned with the initial value problem for non-stationary Stokes flows, under a certain non-linear boundary condition which can be called the leak boundary condition of friction type. Theoretically, our main purpose is to show the strong solvability (i.e.,the unique existence of the L^2-strong solution) of this initial value problem by means of the non-linear semi-group theory originated with Y. Kōmura. The method of analysis can be applied to other boundary or interface conditions of friction type. It should be noted that the result yields a sound basis of simulation methods for evolution problems involving these conditions.

Key words: Stokes equation, Leak boundary condition, Nonlinear semigroup.

1. Introduction

The purpose of this paper is to consider the initial value problem for the Stokes flow under nonlinear boundary conditions of friction type, which will be described below in §2 together with our motivations arising from applications, and to show that the solvability can be obtained immediately by means of the non-linear semigroup theory (NSG theory) which had originated from the celebrated work by Y. Kōmura ([12]) in 1967 and was elaborated by many authors (for a concise explanation of the theory, we refer to Sections 6 and 7 of Chapter XIV of Yosida ([15]).

In order to apply the NSG theory for integration of the initial boundary value problem, a crucial step is to define a multi-valued operator A in a Hilbert space so that A is maximally accretive (m-monotone) and $-A$ plays the role of the generator of the NSG relevant to the initial boundary value problem. Below we shall see that the property of being multi-valued of our generator is closely related with involvement of the pressure in the Stokes equation. This observation might be interesting as a new direction of applicability of the subtle NSG theory. On the other hand, once the NSG theory applies, we could have a better insight into the mathematical structure of the problem, which would lead to a reliable basis in organizing approximating methods. In fact, we have the product formula for the NSG (the solution operator of the problem) as is stated in §5.

Our study of the Stokes equation under the boundary conditions of friction type goes back to the author's series of lectures at Collège de France in October of 1993. Since then, as to the stationary flow, i.e., to the boundary value problem, the existence and uniqueness of the H^1 class solution has been established by means of the formulation through variational inequalities by the author and his collaborators (Fujita [6], Fujita-Kawarada [7], Fujita-Kawarada- Sasamoto [8]). This will be mentioned in §3.

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Recently, Norikazu Saito succeeded in showing the H^2 regularity of the solution of the boundary value problem, following suggestions given by H. Brezis and the author ([14]). His result will be also described in §3.

N. Saito's regularity result has led the author to the definition below of the required generator to match the NSG theory. This and integration of the initial problem with the aid of NSG will be done in §5 after we recall a core part of Komura's theory of NSG in §4.

2. Description of the Target Problem

First of all, let us write down the initial boundary value problem in question (abbr. S-IVP).

Usual symbols are employed: $u = u(t,x)$ and $p = p(t,x)$ stand for the flow velocity and the pressure, respectively, at time t and point x. x ranges over a bounded domain Ω in $R^n (n = 2, 3)$ bounded by smooth boundary $\Gamma = \partial\Omega$. The positive constant ν means the viscosity.

Time-dependent Stokes system

For $t \geq 0$ and $x \in \Omega$, $\{u, p\}$ should satisfy

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t} & = & \nu \Delta u - \nabla p + f, \\ \operatorname{div} u & = & 0, \end{array} \right. \tag{2.1}$$

where f stands for the given external force.

The initial condition is given by

$$u(0, \cdot) = u_0 \qquad \text{in } \Omega. \tag{2.2}$$

In order to avoid non-essential complexity in phrasing, we assume that Γ is composed of two separated compact component Γ_0 and S, and that on Γ_0 the homogeneous Dirichlet boundary condition is imposed; namely,

$$u = 0 \qquad \text{on } \Gamma_0. \tag{2.3}$$

On the other hand, we shall impose

$$\text{a certain boundary condition of friction type} \qquad \text{on } S, \tag{2.4}$$

which will be specified soon.

2.1. Motivations for the BC of Friction Type

So far almost exclusively, the Dirichlet boundary condition (adhesion to solid surfaces) has been considered for motions of viscous incompressible fluids in hydrodynamics as well as in mathematics. However, there exist some flow phenomena, modeling of which might require introduction of slip and/or leak boundary conditions in reality or apparently (or metaphorically).

As examples, we can refer to the following; (1) flow through a drain or canal with its bottom covered by sherbet of mud and pebbles. (2) flow of melted iron coming out from a smelting furnace. (3) avalanche of water and rocks. (4) blood flow in a vein of an arterial sclerosis patient. (5) polymer-polymer welding and sliding phenomena as studied by P.G.de Gennes.

Furthermore, with some of these examples one observes that some fragile state of the surface or existence of sherbet zone *allows slipping of the fluid along the surface*, while *the fluid does not slip as long as the "force of stream" is below a threshold.*

In order to form a mathematical model of such slip phenomena, introduction of nonlinear slip boundary conditions of friction type (similar to Coulomb's law of friction) seems to be suitable.

Similarly, leak boundary conditions would be required when we want to model flow problems involving a leak of the fluid through the surface or penetration into the adjacent media. For instance, (1) flow through a net or sieve, e.g., a butterfly net. (2) flow through filter, e.g., a