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ORTHOGONAL PIECE-WISE POLYNOMIALS BASIS ON AN ARBITRARY TRIANGULAR DOMAIN AND ITS APPLICATIONS*[1)]

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

This paper present a way to construct orthogonal piece-wise polynomials on an arbitrary triangular domain via barycentric coordinates. A boundary value problem for Laplace equation and its eigenvalue problem can be solved as two applications of this approach.

Key words: Orthogonal piece-wise polynomials, Triangular domain, Eigen-decomposition.

1. Introduction

It is well known that the concept of orthogonal polynomials plays a key role in numerical analysis. How to generalize orthogonal polynomials into higher dimension, without using tensor product, is still an open problem. As we know, the original orthogonal polynomial has been studied in univariable case. Strictly, the tensor product approach is still staying in the one dimension level via decreasing dimension. The result only can use in rectangular domain.

There are some different ways to construct orthogonal polynomials in univariable case, such as three-term recurrence, generalized function, tridiagonal matrices, Gram orthogonal procedure, etc. Unfortunately, most of them are only suitable for univariate case. However, so-call Sturm-Liouville approach constructs orthogonal polynomial via eigen-decomposition of an ordinary differential operator with two-points boundary values. The eigen-decomposition approach may extend to high dimension by solving the related partial differential equation. Moreover, in high dimension simplex domains, barycentric coordinates and Bernstein form of multivariate polynomial or B net have essential advantage doing analysis than the usual Cartesian coordinates.

The eigen-decomposition for the Laplace operator in two dimensions is a classical problem in mathematics and physics. Especially for Dirichlet and Neumann boundary conditions, there are many theoretical results on the eigenvalues and eigenfunctions that have some application to numerical methods, e.g. [1]. If there were an effective asymptotic expression for the nth eigenvalue λ_n, only a finite number of the lowest eigenvalues would need to be calculated for any given region.

In practical case, one needs to calculate a lot of eigenvalue problems, such as large scale linear solver arisen from oil-reservoir simulation, preconditioning technique deal with high ill-condition, unstructured mesh in oil-reservoir simulation software, high precision algorithm for solving partial sum of eigenvalues of Schrodinger equation arisen from material science and so on.

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In this paper we consider Laplace operator with Dirichlet boundary condition on a triangular domain Ω as:

$$\begin{cases} -\Delta u = \lambda u, & u \text{ in } \Omega \\ u = 0, & u \text{ on } \partial\Omega \end{cases}$$

where the side h_1, h_2, h_3 of Ω satisfy $h_1 \leq h_2 \leq h_3$.

This problem may represent the vibration of a fixed membrane and the propagation a wave down a waveguide[1,2]. Pinsky[3] presented the exactly formulae for the eigenvalues and eigenfunctions as $h_1 = h_2 = h_3$, and Prager[4] studied them for the 30-60-90 case. In this paper, we discuss an approximate method to calculate the eigenvalues and eigenfunctions of Laplace operator with Dirichlet boundary condition in an arbitrary triangular domain. By using barycentric coordinates and Bernstein polynomials, an orthogonal piece-wise polynomials basis on the triangle via approximating eigen-decomposion are constructed, 4-16 approximating eigenvalues and eigen-vectors for 2-D Laplacian operator on an arbitrary triangle are obtained. As an example, one approximates solution for Poisson equation on a triangle domain is given out at last.

2. Barycentric Coordinates and Bernstein Polynomials on High Dimension Simplex Domain

For a given n-D simplex with $n+1$ vertices

$$\{P_1, P_2, \cdots, P_{n+1}\}$$

the barycentric coordinate system is defined as

$$t = (t_1, t_2, \cdots, t_{n+1}),$$

with

$$t_k = \text{Vol}\,[P_1, P_2, \cdots, P_{k-1}, P, P_{k+1}, \cdots, P_{n+1}]/V$$

where the volume

$$V := \text{Vol}\,[P_1, P_2, \cdots, P_{n+1}].$$

It is obvious that

$$\sum_{k=1}^{n+1} t_k = 1, \qquad 0 \leq t_k \leq 1.$$

Set $k = (k_1, k_2, \cdots, k_{n+1})$ with $k! = k_1!k_2!\cdots k_{n+1}!$, $|k| = k_1 + k_2 + \cdots + k_{n+1}$, $t^k = t_1^{k_1} t_2^{k_2} \cdots t_{n+1}^{k_{n+1}}$.

Then so-called B-form of multivariate polynomials f(t) is defined as

$$f(t) = \sum_{|k|=m} f_k b_k^m(t)$$

where f_k is called B-net of the m-degree polynomial over $n-$ dimension simplex domain, and

$$b_k^m(t) = \frac{m!}{k!} t^k$$

forms the related B-B basis, or Bernstei-Bézier basis.

Especially, $f(t) = 0$ or $f(t) = 1$ if and only if all $f_k = 0$ or all $f_k = 1$, respectively.

If $f(t)$ is a piecewise polynomial with some global continuous conditions, we may represent it in terms of the related piecewise B-net to keep the required continuous conditions, e.g. see [5], [7] or [8].