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OPTIMAL MIXED $H-P$ FINITE ELEMENT METHODS FOR STOKES AND NON-NEWTONIAN FLOW*

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

Based upon a new mixed variational formulation for the three-field Stokes equations and linearized Non-Newtonian flow, an $h-p$ finite element method is presented with or without a stabilization. As to the variational formulation without stabilization, optimal error bounds in h as well as in p are obtained. As with stabilization, optimal error bounds are obtained which is optimal in h and one order deterioration in p for the pressure, that is consistent with numerical results in [9, 12] and therefore solved the problem therein. Moreover, we proposed a stabilized formulation which is optimal in both h and p.

Key words: Mixed hp- finite element method, Non-Newtonian flow, Stabilization, Scaled weak $B-B$ inequality.

1. Introduction

Motivated by some advantages of $h-p$ FEM over the classic FEM uncovered by recent computation works(see-[19]), Schwab and Süri [11] have considered the mixed $h-p$ finite element method for Non-Newtonian flow based upon a three-field Stokes formulation emanating from linearization of some different models of Non-Newtonian flow, in which, stress, velocity and pressure are coupled. Theoretical analysis and tailored numerical experiments show that the mixed $h-p$ finite element method exhibits an exponential convergence on geometrical graded meshes. However, optimal error bounds for both h and p are not available, and extra freedoms are needed for the stress if a continuous approximation is preferable, the latter is momentous when the equation has to be coupled with other equations in a big system or the problem is set up in high-dimensions. Though a modified EVSS method (Elastic Viscous Split Stress) [7] makes the drop of redundant freedoms possible, it gives rise to a non-symmetric system with an extra unknow which increases the complexity of computations.

Combined with the well-known stabilized FEM(see [8] for a survey), Schötzau, Gerdes and Schwab proposed a stabilized $h-p$ FEM in [13] and [9]. However, error bounds obtained therein are not consistent with numerical tests [9, 12]. It seems that such discrepancy is merely due to techniques employed.

The purpose of this paper is developing a unifying method, a stabilized $h-p$ FEM to resolve the above problem. Our method relies on a new variational formulation. The main advantage of this method is that the choice of finite element spaces for the stress is independent of those for the velocity and pressure. The ingredient in our analysis is the *scaled weak B-B inequality* proved in the $h-p$ setting and the *divide and conquer* principle.

Outline of the paper follows. In the next section, a new variational formulation is proposed and a finite element space pair is presented and analyzed. As a direct consequence of this variational formulation, an iterative algorithm is deduced in Section §3, convergence rate is also

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estimated. In the last section, a stabilized $h-p$ FEM is formulated and the error bound is derived which is consistent with numerical results.

Throughout this paper, we assume that the constant C is independent of h and p.

2. New Variational Formulation for Upper Convected Maxwell Model

In the following, we only consider the upper convected Maxwell model, the simplest one in Non-Newtonian fluids, which can be described by the following equations:

$$-\operatorname{div}\boldsymbol{\sigma}+\boldsymbol{\nabla}p=\boldsymbol{f},\qquad \operatorname{div}\boldsymbol{u}=0,\qquad \boldsymbol{\sigma}+\lambda\frac{\delta\boldsymbol{\sigma}}{\delta t}=2\nu\mathcal{E}\boldsymbol{u}, \tag{2.1}$$

where $\mathcal{E}\boldsymbol{u}$ is the strain rate tensor defined by the symmetric part of $\boldsymbol{\nabla u}$ as $\mathcal{E}\boldsymbol{u}=\frac{1}{2}(\boldsymbol{\nabla u}+(\boldsymbol{\nabla u})^T)$, $\delta\boldsymbol{\sigma}/\delta t$ is the upper convected derivative defined by

$$\frac{\delta\boldsymbol{\sigma}}{\delta t}=\frac{\partial\boldsymbol{\sigma}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma}-(\boldsymbol{\nabla u}\cdot\boldsymbol{\sigma}+\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla u})^T). \tag{2.2}$$

λ is the relaxation time of the material. Let $\lambda=0$, (2.1) reduces to

$$-\operatorname{div}\boldsymbol{\sigma}+\boldsymbol{\nabla}p=\boldsymbol{f},\qquad \operatorname{div}\boldsymbol{u}=0,\qquad \boldsymbol{\sigma}=2\nu\mathcal{E}\boldsymbol{u}, \tag{2.3}$$

which is just the three-field Stokes problem.

We introduce some notations.

Let Ω be a bounded convex polygonal domain in $\mathcal{R}^2$ with the Lipschitz boundary Γ. $\mathcal{R}^2$ is equipped with Cartesian coordinates $x_i,\ i=1,2$. Denote by $(\cdot,\cdot)$ the $\mathcal{L}^2(\Omega)$ scalar product of functions, vectors or tensors. Defined the following Sobolev spaces: $\boldsymbol{T}=[\mathcal{L}^2(\Omega)]^4_{\text{sym}}=\{\tau=(\tau_{ij})\mid \tau_{ij}=\tau_{ji},\ \tau_{ij}\in\mathcal{L}^2(\Omega),\ i,j=1,2\}$ with the norm $\|\tau\|_{\boldsymbol{T}}=(\int_\Omega|\tau|^2)^{\frac{1}{2}}$, $\boldsymbol{X}=[H^1_0(\Omega)]^2, M=\mathcal{L}^2_0(\Omega)=\{q\in\mathcal{L}^2(\Omega)\mid\int_\Omega q=0\}$. X,M are equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{X}}=(\int_\Omega|\mathcal{E}\boldsymbol{v}|^2)^{\frac{1}{2}},\ \|q\|_M=(\int_\Omega|q|^2)^{\frac{1}{2}}$ respectively. It is easy to see that $\|\cdot\|_{\boldsymbol{X}}$ is an equivalent norm over $\boldsymbol{X}$.

With these notations, we state a new variational formulation as follows.

Find $(\boldsymbol{\sigma},\boldsymbol{u},p)\in\boldsymbol{T}\times\boldsymbol{X}\times M$ such that

$$\frac{\alpha}{2\nu}(\boldsymbol{\sigma},\boldsymbol{\tau})-\alpha(\boldsymbol{\tau},\mathcal{E}\boldsymbol{u})=0\qquad\forall\boldsymbol{\tau}\in\boldsymbol{T}, \tag{2.4}$$

$$\alpha(\boldsymbol{\sigma},\mathcal{E}\boldsymbol{v})+2(1-\alpha)\nu(\mathcal{E}\boldsymbol{u},\mathcal{E}\boldsymbol{v})-(p,\operatorname{div}\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v})\qquad\forall\boldsymbol{v}\in\boldsymbol{X}, \tag{2.5}$$

$$(\operatorname{div}\boldsymbol{u},q)=0\qquad\forall q\in M. \tag{2.6}$$

Remark 2.1. The above formulation is similar to the Oldroyd version of the Stokes problem [2] with $\nu=1$. However, a finite element discretization of the latter yields a non-symmetric algebraic system while the previous one gives rise to a symmetric system with a variant of α that accounts for the flexibility in applications.

To facilitate the analysis, we define two operators as follows

$$A_\alpha(\cdot,\cdot):\ \boldsymbol{T}\times\boldsymbol{X}\times\boldsymbol{T}\times\boldsymbol{X}\to\mathcal{R},$$

$$A_\alpha(\boldsymbol{\sigma},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v})=\frac{\alpha}{2\nu}(\boldsymbol{\sigma},\boldsymbol{\tau})-\alpha(\boldsymbol{\tau},\mathcal{E}\boldsymbol{u})+\alpha(\boldsymbol{\sigma},\mathcal{E}\boldsymbol{v})+2(1-\alpha)\nu(\mathcal{E}\boldsymbol{u},\mathcal{E}\boldsymbol{v}) \tag{2.7}$$

and

$$B(\cdot,\cdot):\ \boldsymbol{T}\times\boldsymbol{X}\times M\to\mathcal{R},$$

$$B(\boldsymbol{\tau},\boldsymbol{v};q)=-(p,\operatorname{div}\boldsymbol{v}). \tag{2.8}$$

Problem $\boldsymbol{H}$: find $(\boldsymbol{\sigma},\boldsymbol{u},p)\in\boldsymbol{T}\times\boldsymbol{X}\times M$ such that

$$A_\alpha(\boldsymbol{\sigma},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v})+B(\boldsymbol{\tau},\boldsymbol{v};p)=(\boldsymbol{f},\boldsymbol{v})\qquad\forall(\boldsymbol{\tau},\boldsymbol{v})\in(\boldsymbol{T},\boldsymbol{X}), \tag{2.9}$$