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A NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR EXTERIOR 3-D PROBLEM*[1)]

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

In this paper, a nonoverlapping domain decomposition method, which is based on the natural boundary reduction(cf. [4, 13, 15]), is developed to solve the boundary value problem in exterior three-dimensional domain of general shape. Convergence analyses both for the exterior spherical domain and the general exterior domain are made. Some numerical examples are also provided to illustrate the method.

Key words: Domain decomposition, D-N Algorithm, Exterior 3-D problem.

1. Introduction

In recent years, the elliptic boundary value problems in unbounded domains have drawn more and more attention. To solve an equation in an unbounded domain numerically, a basic idea is to limit the computation to a bounded domain by introducing an artificial boundary. Based on this idea, many numerical methods, such as the coupling of BEM and FEM, the FEM with boundary conditions at artificial boundary, the coupled finite-infinite element method, the DDM(domain decomposition method)(cf.,e.g., [7, 6, 12, 3, 5, 2, 16, 17] and so on), have been put forward. All these methods have their own advantages as well as limitations. It is a practicable way to combine the natural boundary element method with the traditional FEM and DDM to solve problems in unbounded domains. However, the methods given in some published papers are only for two-dimensional cases(cf. [16, 17]) and cannot be directly extended to three-dimensional problems. In this paper, by taking Poisson equation as an example, we shall suggest a nonoverlapping DDM for exterior three-dimensional problems. By choosing a sphere as an interface, we turn the original problem into two subproblems, i.e., one in a bounded domain and the other in a regular unbounded domain(exterior spherical domain). We then solve the two subproblems alternately to acquire an approximate solution of the original problem. The subproblem in bounded domain is treated by the traditional FEM. The unique aspect of our method is to adopt the recent results of the natural boundary element method (cf. [10]) to solve the subproblem in unbounded domain, which makes our method simple in analysing and easy to be implemented.

The rest of this paper is organized as follows. Section 2 develops the D-N alternating algorithm; Section 3 studies the convergence of the D-N method for exterior spherical domain; Section 4 extends the result of Section 3 to general exterior domain; Section 5 discusses the discrete form of the D-N alternating algorithm; Section 6 presents some numerical results.

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2. Dirichlet-Neumann(D-N) Alternating Algorithm

Consider the following exterior boundary value problem:

$$\begin{cases} -\triangle u = f, & \text{in } \Omega^c, \\ u = g, & \text{on } \Sigma_0, \end{cases} \tag{2.1}$$

where $\Omega \subset R^3$ is a bounded domain and Ω^c denotes $R^3 \backslash \overline{\Omega}$. $\Sigma_0 = \partial\Omega$ is a piecewise smooth surface. $g \in H^{\frac{1}{2}}(\Sigma_0)$ and $f \in L^2(\Omega^c)$ are given functions. To guarantee ensure the existence and uniqueness of the solution of (2.1), we must assume that u vanishes at infinity(cf. [18]). In the following discussion we shall also assume that function f has compact support.

Introduce a sphere $\Sigma_1 = \{(r, \theta, \varphi) | r = R_1\}$ for an appropriate R_1 to enclose boundary Σ_0 and the support of f. Make sure that $\text{dist}(\Sigma_1,\ \Sigma_0) > 0$. Then Ω^c is decomposed into two mutually disjoint subdomains, i.e., an interior subdomain denoted by Ω_1 and an exterior subdomain denoted by Ω_2. Ω_1 and Ω_2 are nonoverlapping domains. For exterior boundary value problem (2.1), we suggest the following D-N(Dirichlet-Neumann) alternating algorithm:

$$\begin{cases} -\Delta u_2^n = 0, & \text{in } \Omega_2, \\ u_2^n = \lambda^n, & \text{on } \Sigma_1, \\ \lim\limits_{r \to \infty} u_2^n = 0, \end{cases} \tag{2.2a}$$

$$\begin{cases} -\Delta u_1^n = f, & \text{in } \Omega_1, \\ u_1^n = g, & \text{on } \Sigma_0, \\ \dfrac{\partial u_1^n}{\partial n_1} = -\dfrac{\partial u_2^n}{\partial n_2}, & \text{on } \Sigma_1, \end{cases} \tag{2.2b}$$

$$\lambda^{n+1} = \theta_n u_1^n + (1 - \theta_n)\lambda^n, \tag{2.2c}$$

where u_1^n and u_2^n are the n-th approximate solutions in Ω_1 and Ω_2; n_1 and n_2 denote the unit outward normals of Σ_1 with respect to the two neighboring subdomains; θ_n denotes the n-th relaxation factor and λ^0 is an arbitrary function in $H^{\frac{1}{2}}(\Sigma_1)$.

Note that, on interface Σ_1, only the value of the normal derivative of the solution of (2.2a) is needed in solving (2.2b). So it is unnecessary to solve (2.2a). Actually we can obtain $\dfrac{\partial u_2^n}{\partial n_2}$ directly from λ^n by making use of the following natural integral equation(cf. [10]):

$$\frac{\partial u_2^n}{\partial n_2} = -\frac{1}{16\pi R_1} \int_0^{2\pi} \int_0^{\pi} \frac{\lambda^n(\theta',\ \varphi') \sin\theta'}{\sin^3 \frac{\gamma}{2}} d\theta' d\varphi'. \tag{2.3}$$

γ in (2.3) satifies $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$ and the hypersingular integral in (2.3) must be understood as the normalization of divergent integral in the sense of generalized function. The details about the computation of this kind of integrals can be found in [8, 13, 14, 10].

For $\lambda(\theta,\ \varphi) \in H^{\frac{1}{2}}(\Sigma_1)$, make harmonic extensions of $\lambda(\theta,\ \varphi)$ to Ω_1 and Ω_2 respectively to acquire functions $H_1\lambda$ and $H_2\lambda$, namely, $H_1\lambda$ satifies

$$\begin{cases} -\Delta H_1\lambda = 0, & \text{in } \Omega_1, \\ H_1\lambda = \lambda, & \text{on } \Sigma_1, \\ H_1\lambda = 0, & \text{on } \Sigma_0, \end{cases} \tag{2.4}$$

while $H_2\lambda$ satifies

$$\begin{cases} -\Delta H_2\lambda = 0, & \text{in } \Omega_2, \\ H_2\lambda = \lambda, & \text{on } \Sigma_1, \\ \lim\limits_{r \to +\infty} H_2\lambda = 0. \end{cases} \tag{2.5}$$