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THE STABILITY OF LINEAR MULTISTEP METHODS FOR LINEAR SYSTEMS OF NEUTRAL DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the numerical solution of initial value problems for systems of neutral differential equations

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau), y'(t-\tau)) \quad t > 0, \\ y(t) &= \phi(t) \quad t < 0, \end{aligned}$$

where $\tau > 0$, f and ϕ denote given vector-valued functions. The numerical stability of a linear multistep method is investigated by analysing the solution of the test equations $y'(t) = Ay(t) + By(t-\tau) + Cy'(t-\tau)$, where A, B and C denote constant complex $N \times N$-matrices, and $\tau > 0$. We investigate the properties of adaptation of the linear multistep method and the characterization of the stability region. It is proved that the linear multistep method is NGP-stable if and only if it is A-stable for ordinary differential equations.

Key words: Numerical stability, Linear multistep method, Delay differential equations.

1. Introduction

For a large class of electrical networks containing lossless transmission lines the describing equations can be reduced to a system of neutral differential equations.

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$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)) \quad t > 0, \\ y(t) = \phi(t) \quad t < 0, \end{cases} \tag{1}$$

where f and ϕ denote given vector-valued functions with $f(t, x, y, z) \in C^N$ (whenever $t \in \mathbb{R}^+$, $y(t) \in C^N$, $y(t-\tau) \in C^N$), $\phi(t) \in C^N$, $\tau > 0$ and $y(t) \in C^N$is unknown for $t > 0$.

The purpose of the present paper is to investigate the stability properties of the linear multistep methods (LMMs) based on the following test system

$$\begin{cases} y'(t) = Ay(t) + By(t-\tau) + Cy'(t-\tau) \quad t > 0, \\ y(t) = \phi(t) \quad t < 0, \end{cases} \tag{2}$$

where A, B and C denote constant complex $N \times N$-matrices and $\tau > 0$. The solution of (2) is called asymptotically stable if

$$\lim_{t \to \infty} y(t) = 0. \tag{3}$$

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For any matrix X, denote its determinant by $\det[X]$, its spectrum by $\sigma[X]$ and its spectral radius by $\rho[X]$.

In [11], a new and simple criterion on the matrices A, B and C such that the solution of (2) is asymptotically stable has been derived.

Lemma 1.1. *(see [11]) Let $\| C \| < 1$. Then all exact solutions to (2) are asymptotically stable if*

$$\forall \lambda \in \sigma[A] \Longrightarrow \Re(\lambda) < 0, \tag{4}$$

$$\sup_{\Re(\xi)=0} \rho[(\xi I - A)^{-1}(\xi C + B)] < 1, \tag{5}$$

where $\| A \| = \sup_{\|x\|=1} \| Ax \|$, $\| x \|^2 = \langle x, x \rangle$, $x \in C^N$.

Consider the ordinary differential equations

$$\begin{aligned} x'(t) &= f(t, x(t)) \qquad t > 0, \\ x(0) &= x_0, \end{aligned}$$

where $x(t)$ and $f(t, x(t))$ are vector-valued functions. If $h > 0$ denotes a given stepsize, the gridpoint t_n is given by $t_n = nh$, and x_n denotes an approximation to $x(t_n)$. A linear multistep method can be written as

$$\sum_{j=0}^{k} \alpha_j x_{n-j} = h \sum_{j=0}^{k} \beta_j f(t_{n-j}, x_{n-j}) \quad (n = k+1, k+2, \ldots). \tag{6}$$

Here α_j and $\beta_j (j = 0, 1, 2, \cdots, k)$ denote the coefficients of a LMM. Let $\rho(z)$ and $\sigma(z)$ be the usual characteristic polynomials

$$\begin{aligned} \rho(z) &= \sum_{j=0}^{k} \alpha_j z^{k-j}, \\ \sigma(z) &= \sum_{j=0}^{k} \beta_j z^{k-j}. \end{aligned}$$

A linear multistep method (ρ, σ) is called A-stable if all roots z of $\rho(z) - \lambda\sigma(z) = 0$ satisfy $|z| < 1$ whenever $\Re(\lambda) < 0$. Then we easily obtain the following result.

Lemma 1.2. *The linear multistep method is A-stable if and only if $\rho(z)I - \sigma(z)A$ is invertible (whenever all eigenvalues λ of A satisfy $\Re(\lambda) < 0$ and $|z| \geq 1$).*

In section 2, we will present adaptations of the linear multistep methods for the numerical solution of (1). In section 3, we investigate the stability region of the methods with respect to the test system (2) and some equivalent conditions are established.

2. Adaptations of Linear Multistep Methods

In order to adapt the linear multistep methods (6) to (1), we introduce unknowns $u(t)$ and $v(t)$ as

$$u(t) = y(t - \tau) \text{ and } v(t) = y'(t - \tau). \tag{7}$$