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TETRAHEDRAL C^m INTERPOLATION BY RATIONAL FUNCTIONS[*1)]

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Abstract

A general local $C^m (m \geq 0)$ tetrahedral interpolation scheme by polynomials of degree $4m+1$ plus low order rational functions from the given data is proposed. The scheme can have either $4m+1$ order algebraic precision if C^{2m} data at vertices and C^m data on faces are given or $k + E[k/3] + 1$ order algebraic precision if C^k $(k \leq 2m)$ data are given at vertices. The resulted interpolant and its partial derivatives of up to order m are polynomials on the boundaries of the tetrahedra.

Key words: C^m interpolation, Rational functions, Tetrahedra.

1. Introduction

We consider the problem of constructing C^m $(m \geq 0)$ piecewise rational local interpolation to the data on a domain in $I\!R^3$ that is assumed to have been tessellated into tetrahedra (we denote the tessellation by $\mathcal{T}$). The scheme requires the following data: The partial derivatives of order s at each vertex for $s = 0, 1, \cdots, 2m$, partial derivatives of order s at s equally (no necessary) distributed points (excluding the end points) on each edge, and $\frac{1}{2}[(m+2s)(m+2s-1)-3s(s-1)]$ regularly distributed points on each face for $s = 0, \cdots, m$ (see section 4 for detail).

Interpolation over tetrahedra is a fundamental problem in the areas of data fitting, CAGD and finite element analysis. Many schemes have been developed for constructing C^1 interpolants. These schemes can be classified into three categories. The schemes in the first category require the interpolants to be polynomials over the given tetrahedra. In (Rescorla, [2]) a C^1 piecewise polynomial of degree 9 interpolation scheme is presented which needs C^4 data at the vertices. In general, a C^m piecewise polynomial interpolation scheme requires a polynomial of degree $8m+1$ and C^{4m} data (see [6]). It should be noted that this approach needs much higher order of data and higher degree of the polynomial than the order of smoothness that the scheme can achieve. To avoid such disadvantages, subdivision schemes, that may be classified into the second category, are developed. In these schemes, each tetrahedron is split into sub-tetrahedra using Clough-Tocher split (see Alfeld, [2], Worsey and Farin, [8] and Farin, [5]) or Powell-Sabin split (see Worsey and Piper [9]). In (Alfeld, [2]), Clough-Tocher split is used to split each tetrahedron into twelve sub-tetrahedra, and C^2 data and quintic are used to achieve C^1 continuity. An n-dimensional Clough-Tocher scheme is proposed by Worsey and Farin, [8]. In (Worsey and Piper,[9]), each tetrahedron is split into twenty-four sub-tetrahedra, and C^1 data and quadratic are used to achieve C^1 continuity. The main disadvantage of this approach is that it leads to more sub-tetrahedra hence more pieces of functions. For examples, the Clough-Tocher split may cause many thin sub-tetrahedra which may affect the stability of the interpolant. The third category of the schemes use rational form interpolants. The rational interpolants avoid the split of the tetrahedra. In (Alfeld, [1]), a transfinite C^1 scheme is proposed, and through the discretization of the transfinite scheme a finite C^1 rational interpolant is derived. In (Barnhill and Little, [4]), a C^1 BBG interpolant, which is then discretized to a 28-degrees-of-freedom C^1 scheme. However, such a discretization is rather complicated. The most general simplicial

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rational interpolation scheme is the *perpendicular interpolation* described in [3]. The C^m interpolation scheme requires C^m data at vertices, uses rational function with denominator degree $6m+12$ (for even m) or $6m+6$ (for odd m), and has order m or $m+1$ algebraic precision. To achieve the goal of using lower order polynomials, global spline interpolation methods have been proposed by Wang and Shi (see [10]) for constructing C^1 interpolants in any dimension.

In this paper, we shall use the rational form to construct locally C^m interpolant for any integer $m \geq 0$. For achieving global C^m continuity, we require C^{2m} data at the vertices and C^m data on the faces and use a polynomial of degree $4m+1$ plus a rational term with denominator degree at most $3m$. The polynomial part will interpolate up to $n := E[m/2]$ order data, while the rational part, which and its partial derivatives of up to order m are polynomials on the boundary of the tetrahedra, will interpolate higher order data. We should mention that all the parameters appeared in the interpolant in our scheme are linear. Hence the interpolants are not only useful in the CAGD area, but also suitable for the finite element analysis. The fact of the interpolant and its partial derivatives are polynomials do have some advantages. It makes the construction of the interpolant as easy as polynomial. This feature is important in some applications in which only boundary values (including derivatives) are involved. Comparing with the perpendicular interpolation of [3], the advantages of our schemes are: the interpolants use lower order rational functions, achieve higher order algebraic precisions and have polynomial boundary feature. We should point out that although the algebraic precision is not crucial in the area of scattered data interpolation, but it is important in the application of the finite element analysis, since it relates to the convergence order. The disadvantage of our scheme is that more data (face data and C^{2m} vertex data) are involved. However, we propose an approach to obtain these data when only lower order data at vertex are given.

The paper is organized as follows: Section 2 gives the notations and the forms of the rational interpolation functions. Sections 3 shows that the used rational functions are well defined and have the required smoothness and have minimal degree properties. Section 4 establishes the formulas for computing the coefficients of the interpolants. In section 5, we discuss the dimension of the interpolation function space, and in section 6 we consider the algebraic precision that the interpolant can achieve.

2. Interpolation Forms

The interpolants in this paper are locally defined on tetrahedra as trivariate polynomials plus trivariate rational functions. The polynomials used in this paper are in Bernstein-Bezier (BB) forms over tetrahedra. Let $p_i = (x_i, y_i, z_i)^T \in I\!R^3$ for $i = 1, \cdots, 4$. Then the tetrahedron, denoted by $[p_1p_2p_3p_4]$, with vertices p_i is defined by $[p_1p_2p_3p_4] = \{p \in I\!R^3 : p = \sum_{i=1}^4 \alpha_i p_i, 0 \leq \alpha_i \leq 1, \sum_{i=1}^4 \alpha_i = 1\}$ where $(\alpha_1, \cdots, \alpha_4)^T$ is known as barycentric coordinate of p. On a tetrahedron, a trivariate polynomial of degree n is expressed by $f(\alpha) = f(\alpha_1, \cdots, \alpha_4) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha_1, \cdots, \alpha_4)$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T \in Z_+^4$, $|\lambda| = \sum_{i=1}^4 \lambda_i$ and $B_\lambda^n(\alpha_1, \cdots, \alpha_4) = \frac{n!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!}\alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\alpha_4^{\lambda_4}$, where Z_+^4 is the collection of the four dimensional vectors with nonnegative integer components. As a subscript, λ stands for $\lambda_1\lambda_2\lambda_3\lambda_4$ or $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Now we consider the directional derivatives of $f(\alpha)$. If we use the symbolic shift operator E_j, i.e., $E_j b_\lambda = b_{\lambda+e_j}$ for $j = 1, \cdots, 4$, where $e_j = (\delta_{jl})_{l=1}^4$ is the jth unit vector in $I\!R^4$, then $f(\alpha)$ can be expressed as $f(\alpha) = \left(\sum_{i=1}^4 \alpha_i E_i\right)^n b_0$. Let $\xi = (\xi_1, \cdots, \xi_4)^T$ be a directional vector in barycentric coordinate, that is, ξ is the difference of the barycentric coordinates of two points q_1 and q_2 in $I\!R^3$(hence $\sum_{i=1}^4 \xi_i = 0$), then directional derivative $D_\xi f(\alpha) = n\left(\sum_{i=1}^4 \alpha_i E_i\right)^{n-1}\left(\sum_{i=1}^4 \xi_i E_i\right) b_0$. It is not difficult to check that $D_{q_1-q_2}F(p) = D_\xi f(\alpha)$, where $F(p)$ is the Cartesian coordinate form of $f(\alpha)$. More generally, let $\xi_j = (\xi_1^{(j)}, \cdots, \xi_4^{(j)})^T$, $j = 1, 2, ..., s (s \leq n)$ be any s directional vectors, then the s-th order directional derivative is

$$D^s_{\xi_1\xi_2\cdots\xi_s} f(\alpha) = \frac{n!}{(n-s)!}\left(\sum_{i=1}^4 \alpha_i E_i\right)^{n-s} \prod_{j=1}^s \left(\sum_{i=1}^4 \xi_i^{(j)} E_i\right) b_0. \quad (2.1)$$

This equality is used frequently to compute the coefficients of a BB form polynomial around vertices, edges and faces of the given tetrahedron from its partial derivatives.