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# CONSTRAINED RATIONAL CUBIC SPLINE AND ITS APPLICATION*1)

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**Abstract**

In this paper, a kind of rational cubic interpolation function with linear denominator is constructed. The constrained interpolation with constraint on shape of the interpolating curves and on the second-order derivative of the interpolating function is studied by using this interpolation, and as the consequent result, the convex interpolation conditions have been derived.

*Key words*: Rational spline, Constrained design, Constrained interpolation, Convex interpolation, Shape control.

## 1. Introduction

Design of high quality, manufacturable surfaces, such as the outer shape of a ship, car or aeroplane, is an important yet challenging task in today's manufacturing industries. Although significant progress has been made in the last decade in developing and commercializing production quality CAD tools, demand for more effective tools is still high due to the ever increase in model complexity and the needs to address and incorporate manufacturing requirements in the early stage of surface design. Within this content, *constrained design* has been identified as one of the surface design problems that need to be solved [1]. This problem deals with control of the bound of curve/surface, the shape and the curvature in the design process.

Spline interpolation is a useful tool for curve and surface design [2,3,4). But in general, the common spline interpolation, such as B-spline, cubic spline, is a kind of fixed interpolation. It means that the shape of the interpolating curve is fixed for the given interpolating data. If one wishes to modify the shape of the interpolating curve, the interpolating data need to be changed. How the shape of the curve can be modified under the condition that the given data are not changed? In recent years, based on the idea of adding the parameters in the interpolating function, the rational spline have been of interest[6,7,8,9]. But because of those rational cubic splines with quadratic or cubic denominators, it is not convenient to do the tasks such as constraining the interpolating curves to be in the given region or constraining them to be concave or convex. In [10], a rational cubic spline with linear denominators based on function values has been constructed, and it can be used to do the constraint in some case. In this paper, a kind of rational cubic interpolation function with linear denominator based on the function values and the derivatives is constructed. The method proposed is $C^1$ continuous but under certain conditions, a $C^2$ rational spline can be got.

This paper considers the constrained curve interpolation problem both on the shape and the second-order derivative of the interpolant. The sufficient conditions for the interpolating curves to be above (or below) a straight line and/or a quadratic curve in an individual knot interval

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and the necessary and sufficient conditions for constraining the second-order derivative of the interpolating function are derived. In fact, not only necessary and sufficient conditions are given, but the existence conditions for this constrained rational cubic interpolation have been derived also. More interested is that because the second-order derivative of the interpolant function can be controlled by this method, the convex interpolation conditions can be got easily as the consequent result.

This paper is arranged as follows. In Section 2, the general form of the rational cubic spline curves is given, and some of its properties is considered, including the tri-diagonal system of equations for the construction of such a $C^1$ rational cubic curve. Section 3 is about the shape constraint problem. Section 4 is about second-order derivative control problem. The convex interpolation conditions which are the consequent result of the second-order derivative constraint and its existence are considered in Section 5.

## 2. Rational Cubic Interpolation

Let $\{f_i, i = 0, 1, \ldots n\}$ be a given set of data points, where $f_i = f(t_i)$ and $t_0 < t_1 < \cdots < t_n$ is the knot spacing. Also, let $\{d_i, i = 0, 1, \ldots n\}$ denote the first-order derivatives of the being interpolated function $f(t)$ at the knots. Define the $C^1$-continuous, piecewise rational cubic function by

$$P(t)\big|_{[t_i,t_{i+1}]} = \frac{p_i(t)}{q_i(t)}, \tag{1}$$

where

$$\begin{aligned} p_i(t) &= (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1}, \\ q_i(t) &= (1-\theta)\alpha_i + \theta\beta_i, \\ \theta &= (t-t_i)/h_i, \\ h_i &= t_{i+1} - t_i, \end{aligned}$$

and

$$\begin{aligned} V_i &= (2\alpha_i + \beta_i)f_i + \alpha_i h_i d_i, \\ W_i &= (\alpha_i + 2\beta_i)f_{i+1} - \beta_i h_i d_{i+1}, \end{aligned}$$

with $\alpha_i, \beta_i > 0$. $P(t)$ satisfies $P(t_i) = f_i, \quad P'(t_i) = d_i, \qquad i = 0, 1, \cdots, n.$

$P(t)$ is the standard cubic Hermite interpolant if $\alpha_i = \beta_i$. If $d_i, i = 0, 1, \cdots, n$, are not fixed, $P(t)$ can be a $C^2$ rational cubic spline by requiring

$$P''(t_i+) = P''(t_i-)$$

for $i = 1, 2, \cdots, n-1$. This condition leads to the following tri-diagonal system of linear equations:

$$\begin{aligned} & h_i\frac{\alpha_{i-1}}{\beta_{i-1}}d_{i-1} + (h_i(1+\frac{\alpha_{i-1}}{\beta_{i-1}}) + h_{i-1}(1+\frac{\beta_i}{\alpha_i}))d_i + h_{i-1}\frac{\beta_i}{\alpha_i}d_{i+1} \\ = \; & h_{i-1}(1+2\frac{\beta_i}{\alpha_i})\Delta_i + h_i(1+2\frac{\alpha_{i-1}}{\beta_{i-1}})\Delta_{i-1}; i = 1, 2, \cdots, n-1, \end{aligned} \tag{2}$$

where

$$\Delta_i = (f_{i+1} - f_i)/h_i.$$