Journal of Computational Mathematics, Vol.19, No.2, 2001, 151–156.

CONVERGENCE OF (0,1,2,3) INTERPOLATION ON AN ARBITRARY SYSTEM OF NODES*[1)]

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Abstract

Estimations of lower bounds for the fundamental functions of (0,1,2,3) interpolation are given. Based on this result conditions for convergence of (0,1,2,3) interpolation and for Grűnwald-type theorem are essentially simplified and improved.

Key words: Hermite interpolation, Hermite-Fejér interpolation, Convergence.

1. Introduction

This note deals with convergence of (0,1,2,3) interpolation on an arbitrary system of nodes. Fisrt we introduce some definitions and notations.

Let

$$1 \geq x_{1n} > x_{2n} > \cdots > x_{nn} \geq -1, \quad n = 1, 2, ... \tag{1.1}$$

Given a fixed even integer m, let $A_{jk} \in \boldsymbol{P}_{mn-1}$ (the set of polynomials of degree at most $mn-1$) satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \quad j, p = 0, 1, ..., m-1; \; k, q = 1, 2, ..., n. \tag{1.2}$$

Then the (0,1,...,m–1) Hermite-Fejér type interpolation for $f \in C[-1,1]$ is defined by

$$H_{nm}(f, x) := \sum_{k=1}^{n} f(x_k) A_{0k}(x), \quad n = 1, 2, \cdots, \tag{1.3}$$

and the (0,1,...,m–1) Hermite interpolation for $f \in C^{m-1}[-1,1]$ is defined by

$$H_{nm}^{*}(f, x) := \sum_{j=0}^{m-1} \sum_{k=1}^{n} f^{(j)}(x_k) A_{jk}(x), \quad n = 1, 2, \cdots \tag{1.4}$$

(cf. [6]). We also need a well known fact:

$$\|H_{nm}\| := \sup_{\|f\| \leq 1} \|H_{nm}(f)\| = \left\| \sum_{k=1}^{n} |A_{0k}| \right\|, \quad n = 1, 2, \cdots, \tag{1.5}$$

where $\|\cdot\|$ stands for the uniform norm on $[-1,1]$.

Here H_{n2} is the classical Hermite-Fejér interpolation investigated in many papers (cf. [1]). H_{n4} is the so called Krylov-Stayermann interpolation, on which although there have been quite a few papers (cf. [6] and its references), almost all of them consider only interpolation based on the special system of nodes, like zeros of Jacobi polynomials. In this note we will give estimations of lower bounds for the fundamental functions of (0,1,2,3) interpolation. Based on this result conditions for convergence of (0,1,2,3) interpolation and for Grűnwald-type theorem are essentially simplified and improved. We will put these results in the next section and some conjectures in the last section.

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* Received January 15, 1998.
[1)] Project 19671082 Supported by National Natural Science Foundation of China.

2. Results

Let

$$\omega_n(x) = (x-x_1)(x-x_2)...(x-x_n),$$

$$\ell_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}, \quad k=1,2,...,n,$$

$$b_{ik} = \frac{1}{i!}\left[\ell_k(x)^{-m}\right]^{(i)}_{x=x_k}, \qquad j=0,1,...,m-1, \quad k=1,2,...,n,$$

$$B_{jk}(x) = \sum_{i=0}^{m-j-1} b_{ik}(x-x_k)^i, \qquad j=0,1,...,m-1, \quad k=1,2,...,n.$$

Then [4]

$$A_{jk}(x) = \frac{1}{j!}(x-x_k)^j B_{jk}(x)\ell_k(x)^m, \qquad j=0,1,...,m-1, \quad k=1,2,...,n. \tag{2.1}$$

In particular, for $m=4$ we have (cf. [2, (4.1)])

$$\begin{cases} B_{1k}(x) & = [10\ell_k'(x_k)^2 - 2\ell_k''(x_k)](x-x_k)^2 - 4\ell_k'(x_k)(x-x_k)+1, \\ B_{2k}(x) & = -4\ell_k'(x_k)(x-x_k)+1, \\ B_{3k}(x) & = 1. \end{cases} \tag{2.2}$$

The following lemma will play a basic role in this note.

Lemma 1. *Let $m=4$. Then*

$$B_{1k}(x) \ge |B_{2k}(x)|, \quad B_{1k}(x) \ge \frac{1}{2}|B_{3k}(x)| = \frac{1}{2}, \qquad x\in\mathbb{R}, \quad k=1,2,...,n, \tag{2.3}$$

and

$$\sum_{k=1}^n |A_{2k}(x)| \le \frac{1}{2}\sum_{k=1}^n (x-x_k)A_{1k}(x), \quad \sum_{k=1}^n |A_{3k}(x)| \le \frac{2}{3}\sum_{k=1}^n (x-x_k)A_{1k}(x), \qquad x\in[-1,1]. \tag{2.4}$$

Proof. By a well known result (cf. [5, p. 976])

$$\ell_k'(x_k)^2 - \ell_k''(x_k) = \sum_{i\ne k}\frac{1}{(x_k-x_i)^2} > 0$$

it follows from (2.2) that

$$B_{1k}(x) \ge 8[\ell_k'(x_k)(x-x_k)]^2 - 4\ell_k'(x_k)(x-x_k)+1.$$

Using a simple symbol $y := \ell_k'(x_k)(x-x_k)$ we get

$$B_{1k}(x) \ge 8y^2 - 4y + 1$$

and

$$B_{2k}(x) = -4y+1.$$

Hence

$$B_{1k}(x) - B_{2k}(x) \ge 8y^2 \ge 0,$$

$$B_{1k}(x) + B_{2k}(x) \ge 2(2y-1)^2 \ge 0,$$

$$B_{1k}(x) - \frac{1}{2} \ge \frac{1}{2}(4y-1)^2 \ge 0,$$

which is equivalent to (2.3).

By (2.1) and (2.3)

$$\sum_{k=1}^n |A_{2k}(x)| = \frac{1}{2}\sum_{k=1}^n |(x-x_k)^2 B_{2k}(x)\ell_k(x)^4|$$

$$\le \frac{1}{2}\sum_{k=1}^n (x-x_k)^2 B_{1k}(x)\ell_k(x)^4 = \frac{1}{2}\sum_{k=1}^n (x-x_k)A_{1k}(x).$$