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SINE TRANSFORM MATRIX FOR SOLVING TOEPLITZ MATRIX PROBLEMS*[1)]

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Abstract

In recent papers, some authors studied the solutions of symmetric positive definite(SPD) Toeplitz systems $T_n x = b$ by the conjugate gradient method(CG) with different sine transforms based preconditioners. In this paper, we first discuss the properties of eigenvalues for the main known circulant, skew circulant and sine transform based preconditioners. A counter example shows that E.Boman's preconditioner is only positive semi-definite for the banded Toeplitz matrix. To use preconditioner effectively, then we propose a modified Boman's preconditioner and a new Cesaro sum type sine transform based preconditioner. Finally, the results of numerical experimentation with these two preconditioners are presented.

Key words: Preconditioner, Toeplitz systems, The fast sine transform, Conjugate gradient algorithm.

1. Introduction

Strang[1] first studied the use of circulant matrices $\mathbf{C}$ for solving systems of linear equations $T_n x = b$ with

$$T_n := T(t_0, t_1, \cdots, t_{n-1}) = \begin{pmatrix} t_0 & t_1 & \cdots & \cdots & t_{n-1} \\ t_1 & t_0 & t_1 & & \vdots \\ \vdots & t_1 & \ddots & \ddots & \vdots \\ t_{n-2} & \cdots & \cdots & t_0 & t_1 \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix} \tag{1.1}$$

a symmetric positive definite Toeplitz matrix.Numerous authors such as T.Chan[2], R.Chan,etc.[3],[4],[5], Tyrtyshnikov[6], Huckle[7] and T.Ku and C.Kuo[8] proposed different families of circulant/skew-circulant preconditioners.

Appling the preconditioned conjugate gradient algorithm(PCGA) to solve the systems $T_n x = b$, we must find a preconditioner $\mathbf{P}$ such that $\mathbf{P}\, y = d$ can be solved very fast and the eigenvalues of $\mathbf{P}^{-1}T_n$ are clustered around the point one. For the circulant and skew circulant $\mathbf{P}$,$\mathbf{P}\, y = d$ can be solved in $O(n \log n)$ operations by the fast Fourier transform(FFT). To avoid complex arithmetic, R.Chan,Ng and Wong[10] and E.Boman and Koltrach[11] presented two kinds of sine transform based preconditioners $S(T_n)$ and P_n respectively. For Toeplitz matrix with the bandwidth $2\beta + 1$, sine transform based preconditioners can also keeps banded,only $O(\beta^2 n) + O(\beta n) = O(n)$ operations is required per each iterative step when β is a constant independent of n. Since circulant/skewcirculant PCGA costs $O(n \log n)$ operations by the FFT algorithm, this implies that the complexity for sine transform based PCGA is reduced by an

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order of the magnitude compared with complexity of circulant/skewcirculant based the PCGA. Furthermore, numerical results given in [10],[11] show that the convergence performance of these two sine transform based preconditioners is better in terms of the number of iterations than that of the optimal circulant preconditioner. However, the known sine transform based preconditioner sometimes fails, hence the purpose of this paper is to seek a more effective sine transform based preconditioner.

This paper is organized as follows. In section §2 we first study the relationship between the eigenvalues of the main known circulant, skew circulant, sine transform based preconditioners and its Fourier series sums of generating function $f(x)$. For the banded Toeplitz matrix,we prove Boman's preconditioner$\mathbf{P}_n$ is only positive semi-definite, a counter example shows that preconditioner $\mathbf{P}_n$ will fail when $f(\frac{l\pi}{n+1}) = 0$ for some $1 \leq l \leq n$. This fact illustrates the conclusion in [11] that preconditioner $\mathbf{P}_n$ is positive definite is wrong. Since $S(T_n)$ may be positive semi-definite(R.Chan,etc.[10] only proved for sufficiently large n,$S(T_n)$ is positive definite with generating function $f(x) \geq m > 0$). Hence, in section §3 we first present a modified sine transform based preconditioner $\tilde{P}_n$ of $\mathbf{P}_n$,then we develop a Fourier series partial sum type and Cesàro sum sine transform based preconditioners P_N and $S_N(T_n)$ respectively.In section §4 we study the clustering properties for various sine transform based preconditioners. Finally, we present some numerical experimentations confirming our theoretical results.

2. The Eigenvalues of Various Preconditioners

Let us begin to introduce a real function $f(x)$ related to infinite Toeplitz matrx T_∞,namely

$$f(x) = \sum_{k=-\infty}^{+\infty} t_k \exp\{ikx\}, i = \sqrt{-1}, x \in [0, 2\pi] \tag{2.1}$$

The partial sums and Cesàro sums are defined by

$$f_n(x) = \sum_{k=-n}^{n} t_k \exp\{ikx\}, \sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} f_n(x), x \in [0, 2\pi] \tag{2.2}$$

respectively. For symmetric Toeplitz matrix,$t_k = t_{-k}$.

Let S_n stands for the discrete sine transform matrix

$$S_n = \sqrt{\frac{2}{n+1}}(\sin(\frac{ij\pi}{n+1}))_{i,j=1}^{n} \tag{2.3}$$

define the set

$$B_{n\times n} = \{B \in R^{n\times n} : S_n B S_n \text{ is a diagonal matrix}\} \tag{2.4}$$

If $T_n = T(t_0, t_1, \cdots, t_{n-1})$ is a Toeplitz matrix defined as Eq.(1.1), we indicate $H(T_n)$ as the following $n \times n$ matrix

$$H(T_n) = \begin{pmatrix} t_2 & \cdots & t_{n-1} & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ & & & & & 0 \\ t_{n-1} & & & & & t_{n-1} \\ 0 & \ddots & & \ddots & & \vdots \\ \vdots & \ddots & & & & \\ 0 & \cdots & 0 & t_{n-1} & \cdots & t_2 \end{pmatrix} \tag{2.5}$$