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THE RELAXING SCHEMES FOR HAMILTON-JACOBI EQUATIONS*[1)]

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Abstract

Hamilton-Jacobi equation appears frequently in applications, e.g., in differential games and control theory, and is closely related to hyperbolic conservation laws[3, 4, 12]. This is helpful in the design of difference approximations for Hamilton-Jacobi equation and hyperbolic conservation laws. In this paper we present the relaxing system for Hamilton-Jacobi equations in arbitrary space dimensions, and high resolution relaxing schemes for Hamilton-Jacobi equation, based on using the local relaxation approximation. The schemes are numerically tested on a variety of 1D and 2D problems, including a problem related to optimal control problem. High-order accuracy in smooth regions, good resolution of discontinuities, and convergence to viscosity solutions are observed.

Key words: The relaxing scheme, The relaxing systems, Hamilton-Jacobi equation, Hyperbolic conservation laws.

1. Introduction

We are interested in the numerical approximation of viscosity solution of the following first-order Hamilton-Jacobi equation

$$\phi_t + H(\phi_{x_1}, \phi_{x_2}, ..., \phi_{x_d}) = 0, \tag{1.1}$$

with initial data $\phi(x,0) = \phi_0(x)$. It is well known that the solutions to problem (1.1) typically are continuous (typically they are locally Lipschitz continuous) but with discontinuous derivatives, even though the initial data $\phi_0 \in C^\infty$. The nonuniqueness of such solutions to (1.1) also necessitates the introduction of the notions of entropy conditions and viscosity solutions, to pick out a unique practically relevant solution (We refer the reader to [1] for details). Therefore, the numerical schemes for solving (1.1) are expected to have: (i) higher order accuracy; (ii) no spurious oscillations in the presence of discontinuous derivatives.

Hamilton-Jacobi equation is often encountered in applications, e.g., in differential games and control theory, and are closely related to a hyperbolic conservation laws[3, 4, 5, 13]

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial f_i(u)}{\partial x_i} = 0. \tag{1.2}$$

In fact, for the case $d = 1$, (1.1) is equivalent to (1.2) if let $u = \phi_x$ [3]. For $d > 1$, this direct correspondence disappears, but in some sense we can still think about (1.1) as (1.2) "integrated once". This is helpful in the design of difference approximations. For example, successful

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numerical methodology for equation (1.2) should be applicable to equation (1.1)[13], and numerical schemes for equation (1.2) can also be designed from equation (1.1)[3, 18]. Crandal and Lions in [5] discussed an important class of numerical schemes for (1.1), the class of monotone schemes. They also proved convergence of monotone schemes to the viscosity solutions of (1.1). Unfortunately, monotone schemes are at most first order accuracy. In [13] Osher and Shu generalized ENO schemes for (1.2) to (1.1). Computational results have shown good accuracy in regions of smoothness and sharp resolution of discontinuities in the derivatives are obtained. However, implementation of ENO schemes seem to be more inconvenient.

In this paper we will present a class of high resolution relaxing schemes for Hamilton-Jacobi equation, based on using the local relaxation approximation [8, 15, 16, 17]. The relaxing scheme is obtained in the following way: Firstly a linear hyperbolic system with a stiff source term is constructed to approximate the original equation (1.1) with a small dissipative correction. Then this linear hyperbolic system can be solved easily by underresolved stable numerical discretizations. The main advantage of the schemes is to use neither nonlinear or linear Riemann solvers spatially nor nonlinear system of algebraic equations solvers temporally. Moreover, there is no exact or numerical integration in current schemes. The schemes are numerically tested on a variety of 1D and 2D problems, including a problem related to control optimization. High-order accuracy in smooth regions, good resolution of discontinuities in the derivatives, and convergence to viscosity solutions are also shown.

The paper is organized as follows. In section 2, the relaxing systems with a stiff source term are introduced to approximate the equation (1.1). In section 3, the relaxing schemes are constructed. The schemes are shown to have correct asymptotic limit as $\epsilon \to 0^+$. In section 4, some numerical tests are presented on a variety of 1D and 2D problems, including a problem related to control optimization. We conclude the paper with a few remarks in section 5.

We point out here that when the first version of our preprint was completed, Prof. Jin kindly informed the author that in an independent work [private communication], he and Xin considered the numerical passages from systems of conservation laws to Hamilton-Jacobi equations. From their note [9], it is clear that, besides totally different techniques, the results of the two works for relaxing schemes for Hamilton-Jacobi equation are different. Our result is also valid for Hamilton-Jacobi equation in arbitrary space dimensions.

2. The Relaxing Systems for Hamilton-Jacobi Equations

In this section we introduce the relaxing system with a stiff source, to approximate the equation (1.1). For the sake of simplicity in the presentation, we will focus on the single equation. First we consider Hamilton-Jacobi equation in one space variable.

To approximate the equation (1.1)($d = 1$), we can introduce a linear system with a stiff source term (hereafter called the *relaxing system*) as follows:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{1}{\epsilon}(v - H(u)), \end{aligned} \tag{2.1}$$

where the small positive parameter ϵ is the relaxation rate, and a is a positive constant satisfying

$$\mid H'(u) \mid \leq \sqrt{a}, \text{ for all } u \in \mathcal{R}. \tag{2.2}$$

Remark. Here we can also consider the more general $a(x,t)$ instead of the above constant a. The results in this paper are not limited by the above constant a.

System (2.1) is equivalent to the one-dimensional perturbed equation

$$\phi_t + H(\phi_x) = \epsilon(a\phi_{xx} - \phi_{tt}),$$

if we take

$$v = -\phi_t, \; u = \phi_x.$$