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D-CONVERGENCE OF RUNGE-KUTTA METHODS FOR STIFF DELAY DIFFERENTIAL EQUATIONS*[1)]

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Abstract

This paper is concerned with the numerical solution of delay differential equations(DDEs). We focus on the error behaviour of Runge-Kutta methods for stiff DDEs. We investigate D-convergence properties of algebraically stable Runge-Kutta methods with three kinds of interpolation procedures.

Key words: Delay Differential equations, Runge-Kutta methods, D-convergence.

1. Introduction

When considering the applicability of numerical methods for the solution of the delay differential equation (DDE) $y'(t) = f(t, y(t), y(t-\tau))$, it is necessary to analyze the error behaviour of the methods. In fact, many papers have investigated the local and global error behaviour of DDE solvers (cf.[1,2,14]). These error analyses are based on the assumption that the function $f(t, y, z)$ satisfies Lipschitz conditions in both the last two variables. They are suitable for nonstiff DDEs because the Lipschitz constants are moderate-sized. However, they can not be applied to stiff DDEs. For example, consider Hutchinson's equation (cf.[9])

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = a\frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)[1-u(x,t-\tau)], \quad t>0, x\in(0,1), \\ u(x,t)=\phi(x,t), \quad t\in[-\tau,0], x\in(0,1), \\ u(0,t)=u(1,t)=0, t\geq-\tau, \end{cases} \tag{1.1}$$

where $a>0$ is the diffusion coefficient, $\phi(x,t)$ is continuous. We transform the partial DDE (1.1) into a system of ordinary DDE by discretising the space variable x into $(N+2)$ discrete values $(N>0)$, with a constant stepsize in space, $\Delta x = 1/(N+1)$, so that $x_j = j\Delta x, j = 0, 1, \cdots, N+1$. Using the standard central difference operator to approximate the Laplacian we obtain a system with

$$f(t,y(t),y(t-\tau)) = \frac{a}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ \vdots \\ y_N(t) \end{bmatrix} + \begin{bmatrix} y_1(t)(1-y_1(t-\tau)) \\ y_2(t)(1-y_2(t-\tau)) \\ \vdots \\ \vdots \\ y_N(t)(1-y_N(t-\tau)) \end{bmatrix}, \tag{1.2}$$

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where $y_j(t)$ denotes the approximation to $u(x_j,t), j=1,2,\cdots,N$. In this case, the Lipschitz constant L of the function $f(t,y,z)$ with respect to y will contain negative powers of the meshwidth Δx in space. As a consequence, L will be very large for fine space grids, and the error estimates based on L are not realistic. On the other hand, the one-sided Lipschitz constant α is only moderate. Hence estimates based on α are often considerably more realistic than that based on L. In fact, Frank et al. introduced the concept of B-convergence for Runge-Kutta methods applied to stiff ODEs, and established the following basic criteria (cf.[6,7,8])

algebraic stability + diagonal stability + stage order p ⇒ B-convergence with order p.

Burrage and Hundsdorfer [4] further discussed the conditions which guarantee that a Runge-Kutta method has order one higher than the stage order. Li [13] further extended these studies to general linear methods and to initial value problems in Hilbert spaces and established a more efficient theory. Recently, the concept of D-convergence [16] for DDEs, which is a generalization of the concept of B-convergence, was introduced. Zhang and Zhou [16] discussed D-convergence of a class of Runge-Kutta methods, and some first and second order D-convergent methods were found. We proved in [10] that the order of D-convergence equals the consistent order in classical sense for A-stable one-leg methods with linear interpolation. In this paper, we further discuss D-convergence of algebraically stable Runge-Kutta methods. We will discuss D-convergence of general linear methods in other paper.

2. Runge-Kutta Methods for DDEs

Let $\langle\cdot,\cdot\rangle$ be an inner product on C^N and $\|\cdot\|$ the corresponding norm. Consider the following nonlinear equation

$$\begin{cases} y'(t)=f(t,y(t),y(t-\tau)), & t\geq 0,\\ y(t)=\phi_1(t), & t\leq 0, \end{cases} \tag{2.1}$$

where τ is a positive delay term , ϕ_1 is a continuous function, and $f:[0,+\infty)\times C^N\times C^N\to C^N$, is a given mapping which satisfies the following conditions:

$$\mathrm{Re}\langle u_1-u_2, f(t,u_1,v)-f(t,u_2,v)\rangle \leq \alpha\|u_1-u_2\|^2, \quad t\geq 0, u_1,u_2,v\in C^N, \tag{2.2}$$

$$\|f(t,u,v_1)-f(t,u,v_2)\|\leq \beta\|v_1-v_2\|, \quad t\geq 0, u,v_1,v_2\in C^N, \tag{2.3}$$

where α and β are real constants. In order to make the error analysis feasible, we always assume that the problem (2.1) has a unique solution $y(t)$ which is sufficiently differentiable and satisfies

$$\|\frac{d^iy(t)}{dt^i}\|\leq M_i.$$

Remark 2.1. When $\beta=0$, the above problem class has been used widely in stiff ODEs field (cf.[5,12]).

Now we consider the adaptation of Runge-Kutta methods to (2.1). Let (A,b,c) denote a given Runge-Kutta method with $s\times s$ matrix $A=(a_{ij})$ and vectors $b=(b_1,\cdots,b_s)^T, c=(c_1,\cdots,c_s)^T$. In this paper we always assume that $0\leq c_i\leq 1 (i=1,\cdots,s)$. Let $h>0$ be a given stepsize and $y_0=\phi_1(0)$. Define gridpoints $t_n (n=0,1,2,\cdots)$ by $t_n=nh$. Then approximation y_{n+1} to $y(t_{n+1}) (n=0,1,2,\cdots)$ are defined by

$$Y_i^{(n)}=y_n+h\sum_{j=1}^{s}a_{ij}f(t_n+c_jh, Y_j^{(n)}, \bar{Y}_j^{(n)}), \quad i=1,\cdots,s, \tag{2.4a}$$

$$y_{n+1}=y_n+h\sum_{j=1}^{s}b_jf(t_n+c_jh, Y_j^{(n)}, \bar{Y}_j^{(n)}). \tag{2.4b}$$

The argument $\bar{Y}_j^{(n)}$ is defined by $\bar{Y}_j^{(n)}=\phi_1(t_n+c_jh-\tau)$ (whenever $t_n+c_jh-\tau\leq 0$), and denotes an approximation to $y(t_n+c_jh-\tau)$ (whenever $t_n+c_jh-\tau>0$) which is obtained by