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ON THE ESTIMATIONS OF BOUNDS FOR DETERMINANT OF HADAMARD PRODUCT OF H-MATRICES*[1)]

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Abstract

In this paper, some estimations of bounds for determinant of Hadamard product of H-matrices are given. The main result is the following: if $A = (a_{ij})$ and $B = (b_{ij})$ are nonsingular H-matrices of order n and $\prod_{i=1}^{n} a_{ii}b_{ii} > 0$, and A_k and $B_k, k = 1, 2, \cdots, n$, are the $k \times k$ leading principal submatrices of A and B, respectively, then

$$\det(A \circ B) \geq |a_{11}b_{11}| \prod_{k=2}^{n} \Big[|b_{kk}| \frac{\det \mathcal{M}(A_k)}{\det \mathcal{M}(A_{k-1})} + \frac{\det \mathcal{M}(B_k)}{\det \mathcal{M}(B_{k-1})} \Big(\sum_{i=1}^{k-1} \Big| \frac{a_{ik}a_{ki}}{a_{ii}} \Big| \Big) \Big],$$

where $\mathcal{M}(A_k)$ denotes the comparison matrix of A_k.

Key words: H-matrix, Determinant, Hadamard product.

1. Introduction

Let $R^{m\times n}$ denote the set of $m \times n$ real matrices, S_n^+ denote the set of $n \times n$ positive definite real symmetric matrices. For $A = (a_{ij})$ and $B = (b_{ij}) \in R^{m\times n}$, the Hadamard product of A and B is defined as an $m \times n$ matrix denoted by $A \circ B : (A \circ B)_{ij} = a_{ij}b_{ij}$.

We write $A \geq B$ if $a_{ij} \geq b_{ij}$ for all i, j. A real $n \times n$ matrix A is called a nonsingular M-matrix if $A = sI - B$ satisfied: $s > 0, B \geq 0$ and $s > \rho(B)$, the spectral radius of B, let M_n denote the set of all $n \times n$ nonsingular M-matrices. Suppose $A \in R^{n\times n}$, its comparison matrix $\mathcal{M}(A) = (m_{ij})$ is defined by the following:

$$m_{ij} = \begin{cases} |a_{ij}|, & \text{if } i = j \\ -|a_{ij}|, & \text{if } i \neq j \end{cases} \tag{1}$$

A real (or complex) $n \times n$ matrix A is called an H-matrix if its comparison matrix $\mathcal{M}(A)$ is a nonsingular M-matrix, let H_n denote the set of all $n \times n$ nonsingular H-matrices.

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On the estimations of bounds for determinant of Hadamard product of matrices, we have the following well-known result.

Oppenheim's inequality: If $A = (a_{ij})$ and $B = (b_{ij}) \in S_n^+$ then

$$\det(A \circ B) \geq \Big(\prod_{i=1}^{n} a_{ii}\Big) \cdot \det(B) \tag{2}$$

Lynn[2] had proved that inequality (2) holds for M-matrices and Fielder and Ptak[3] given a similar result when A is an M-matrix and B is a weakly diagonally dominant matrix. Jianzhou Liu and Li Zhu[1] improved Oppenheim's inequality recently as following theorem:

Theorem 1[1]**.** *If $A = (a_{ij})$ and $B = (b_{ij})$ are nonsingular M-matrices, A_k and B_k, $k = 1, 2, \cdots, n-1$, are the $k \times k$ leading principal submatrices of A and B, respectively, then*

$$det\ (A \circ B) \geq a_{11} b_{11} \prod_{k=2}^{n} \Big[b_{kk} \frac{det(A_k)}{det(A_{k-1})} + \frac{det(B_k)}{det(B_{k-1})} \Big(\sum_{i=1}^{k-1} \frac{a_{ik} a_{ki}}{a_{ii}} \Big) \Big] \tag{3}$$

In this paper, we shall generalize Jianzhou Liu's results and give an inequality similar to (3) for nonsingular H-matrices.

2. Some Lemmas

In this section, we shall give some lemmas which shall be used in the following.

Lemma 1[4]**.** *If A and $B \in M_n$ then $\mathcal{M}(A \circ B) \in M_n$.*

Lemma 2. *If A and $B \in H_n$ then $A \circ B \in H_n$.*

Proof. By the definition of Hadamard product and the definition of comparison matrix, we can easily obtain the following equality:

$$\mathcal{M}(A \circ B) = \mathcal{M}(\mathcal{M}(A) \circ \mathcal{M}(B)) \tag{4}$$

If A and $B \in H_n$ then $\mathcal{M}(A)$ and $\mathcal{M}(B) \in M_n$ and $\mathcal{M}(\mathcal{M}(A) \circ \mathcal{M}(B)) \in M_n$ by Lemma 1, that is: $\mathcal{M}(A \circ B) \in M_n$. So $A \circ B \in H_n$.

Lemma 3[5]**.** *Let $A = (a_{ij}) \in R^{n \times n}$ with $a_{ij} \leq 0$ for all $i \neq j; i, j = 1, 2, \cdots, n$, then the following conditions are equivalent:*

1. *A is a nonsingular M-matrix.*
2. *A has all positive diagonal elements, and there exists a positive diagonal matrix D such that AD is strictly diagonally dominant.*
3. *All of the leading principal minors of A are positive.*

From the definition of H-matrix and Lemma 3, we can easily prove the following result.

Lemma 4. *A matrix A is nonsingular H-matrix if and only if there exists a positive diagonal matrix D such that AD is strictly diagonally dominant.*

Lemma 5[1]**.** *If A is a strictly diagonally dominant matrix with $a_{ii} > 0, i = 1, 2, \cdots, n$, then*

$$det A \geq det \mathcal{M}(A) > 0$$