Journal of Computational Mathematics, Vol.19, No.4, 2001, 371–384.

ON ENTROPY CONDITIONS OF HIGH RESOLUTION SCHEMES FOR SCALAR CONSERVATION LAWS*1)

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Abstract

In this paper a kind of quadratic cell entropy inequalities of second order resolution SOR-TVD schemes is obtained for scalar hyperbolic conservation laws with strictly convex (concave) fluxes, which in turn implies the convergence of the schemes to the physically relevent solution of the problem. The theoretical results obtained in this paper improve the main results of Osher and Tadmor [6].

Key words: Entropy condition, High resolution schemes, Conservation laws.

1. Introduction

Let us consider the Cauchy problems for nonlinear hyperbolic scalar conservation laws:

$$\begin{aligned} \frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x} &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.1}$$

where the function $f(u) \in C^2(R)$ and the initial data $u_0 \in BV(R)$. As is well known, this problem in general does not admit smooth solution, so that weak solution in the sense of distributions must be considered. Moreover, an entropy condition must be added in order to ensure the uniqueness of the weak solutions.

The research of numerical methods for solving the equation (1.1) has been developed rapidly in this decade. Since appearance of the concept of TVD (total variation diminishing) schemes, various high resolution schemes (TVD, TVB (total variation bounded), ENO (essentially non-oscillatory)) have been applied successfully to computational fluid dynamics. See [3, 4, 7]. The convergence of numerical methods for hyperbolic conservation laws depends on the discrete entropy condition and total variation stability of difference schemes, which has been investigated by many authors (see [1, 2, 5, 6, 8, 9, 10]). Especially, Osher and Tadmor in [6] discussed the convergence and cell entropy inequalities of a class of second order resolution TVD (SOR-TVD) schemes. They introduced a kind of modified flux functions for estimating entropy production, and made use of the Godunov numerical flux to obtain an entropy estimate of the general TVD schemes for the new conservation laws with the modified flux. Moreover, they constructed a class of SOR-TVD schemes and analysed the cell entropy conditions of the SOR-TVD schemes with upwind building block, which results in the convergence of the numerical solutions to the unique physical relevant solution. However, there are two points of their arguments should be noticed.

* Received November 25, 1997; Final revised March 10, 1999.
1) The project supported partly by National Natural Science Foundation No.19901031, State Major Key Project for Basic Research.

Firstly, the modified flux functions introduced in [6] are piecewise linear functions. However the smoothness of the modified flux functions is essentially used in the entropy estimatation ([6], section 6). Secondly, in [6] the authors used the first order accurate upwind scheme as the building block for constructing SOR-TVD schemes and as is well known that the upwind scheme may admit entropy voilating solutions, so, some kind of artificial viscousity terms should be added to the upwind scheme for entropy satisfaction.

In this paper, we present a more direct method of proof, which not only avoids using the nonsmooth modified flux function but also simplifies the method of proving and improves the estimates of entropy production for general TVD schemes. In order to avoid adding artificial viscousity, instead of using upwind building block, we choose the Godunov scheme as the building block and obtained the condition for the SOR-TVD properties of the schemes with this Godunov building block, which implies the convergence of the numerical solution to the unique physical relevant solution of the conservation laws (1.1). Our approach seems more natural comparing with the Osher-Tadmor's arguments.

The outline of the paper is as follows. Section 2 reviews the main elements of the general theory of TVD schemes [8,6]. Section 3 is the heart of the paper, we give the discrete entropy estimates of the general TVD schemes by using a more directly method of proving. In Section 4, based on the Godunov building block, we obtain the entropy inequality of a class of SOR-TVD schemes for strict convex (or concave) conservation laws, which results in the convergence of the SOR-TVD schemes to the unique physical relevant solution.

2. TVD Schemes

Consider the following conservative finite difference schemes of (1.1)

$$u_j^{n+1} = u_j - \lambda(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) \tag{2.1}$$

where $\lambda = \Delta t/\Delta x$ is the mesh ratio, and Δt and Δx the variable mesh size in time and space directions, respectively. $h_{j+\frac{1}{2}}$ denotes the Lipschitz continuous numerical flux

$$h_{j+\frac{1}{2}} = h(u_{j-s+1}, ..., u_{j+s}) \tag{2.2}$$

consistent with the differential one,

$$h(w, ..., w) = f(w) \quad . \tag{2.3}$$

We also assume that the scheme (2.1) can be written in an incremental form

$$u_j^{n+1} = u_j + C^+_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} - C^-_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}} \tag{2.4}$$

where $\Delta u_{j+\frac{1}{2}} = u_{j+1} - u_j$.

From (2.1) and (2.4), we have

$$h_{j+\frac{1}{2}} + \frac{1}{\lambda}C^+_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} = h_{j-\frac{1}{2}} + \frac{1}{\lambda}C^-_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}} \ . \tag{2.5}$$

Let the modified flux

$$\begin{aligned} g_j &= h_{j\pm\frac{1}{2}} + \tfrac{1}{\lambda}C^{\pm}_{j\pm\frac{1}{2}}\Delta u_{j\pm\frac{1}{2}} \\ &= \tfrac{1}{2}\left[h_{j-\frac{1}{2}} + h_{j+\frac{1}{2}} + \tfrac{1}{\lambda}C^-_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}} + \tfrac{1}{\lambda}C^+_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}}\right] . \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) one can get

$$C^-_{j+\frac{1}{2}} - C^+_{j+\frac{1}{2}} = \lambda\frac{\Delta g_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}} \ . \tag{2.7}$$