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EXTRAPOLATION AND A-POSTERIORI ERROR ESTIMATORS OF PETROV-GALERKIN METHODS FOR NON-LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*[1)]

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Abstract

In this paper we will show that the Richardson extrapolation can be used to enhance the numerical solution generated by a Petrov-Galerkin finite element method for the initial-value problem for a nonlinear Volterra integro-differential equation. As by-products, we will also show that these enhanced approximations can be used to form a class of a-posteriori estimators for this Petrov-Galerkin finite element method. Numerical examples are supplied to illustrate the theoretical results.

Key words: Volterra integro-differential equations, Petrov-Galerkin finite element methods, Asymptotic expansions, Interpolation post-processing, A-posteriori error estimators.

1. Introduction

The purpose of this paper is to show that the Richardson extrapolation can be used to enhance the numerical solutions generated by a class of Petrov-Galerkin finite element methods for the nonlinear Volterra integro-differential equation (VIDE):

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s))ds, \quad t \in I := [0, T], \quad y(0) = 0, \tag{1.1}$$

where $f = f(t, y) : I \times R \to R$ and $k = k(t, s, y) : D \times R \to R$ (with $D := \{(t, s) : 0 \le s \le t \le T\}$) denote given functions.

Throughout this paper, it will always be assumed that the problem (1.1) possesses a unique solution $y \in C^1(I)$, namely, the given functions $f(t, y)$ and $k(t, s, y)$, which are, respectively, continuous for $t \in I$ and $(t, s) \in D$, will be subject to the following (uniform) Lipschitz conditions [3]:

$$\begin{aligned} &(V1) \quad |f(t, y_1) - f(t, y_2)| \le L_1|y_1 - y_2|, \\ &(V2) \quad |k(t, s, y_1) - k(t, s, y_2)| \le L_2|y_1 - y_2|, \end{aligned}$$

for all $t \in I$, $(t, s) \in D$, and $|y_i| < \infty$ $(i = 1, 2)$.

The Volterra integro-differential equation (1.1) plays an important role in the mathematical modeling of various physical and biological phenomena, and the study of various numerical

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methods has received considerable attention in the past, see, for example, the survey paper [2] by Brunner and the monograph [3] by Brunner et al., as well as references cited therein. Also, see [7] and [16] for some related recent publications.

It is well known that the extrapolation method is a quite effective numerical method in producing more accurate approximations. This technique has been well demonstrated in its applications to the numerical solutions for the elliptic partial differential equations in [1, 5, 10, 11], the parabolic partial differential and integro-differential equations in [6, 8, 12]. This method has also been discussed for boundary element approximations [17, 18]. In [19] multi-parameter parallel algorithms have been introduced into the extrapolation for accelerating the computational speeds.

Here, we shall investigate the extrapolation of the numerical solutions generated by a class of Petrov-Galerkin finite element (PGFE) methods [9] for the initial value problem of a nonlinear VIDE (1.1). The Petrov-Galerkin finite element method has an advantage over the standard Galerkin finite element method, a special case of the Petrov-Galerkin finite element method, in that it allows the trial and test function spaces to be different. This feature provides us with a freedom in choosing a pair of trial and test function spaces for a better computational efficiency than the standard Galerkin finite element method. Moreover, our analysis reveals that the PGFE solutions have a rich collection of approximation properties, and a variety of post-processing techniques have been developed to take advantage of them [15]. In particular, we will show that the Richardson extrapolation can be used to efficiently generate better approximations from the PGFE solutions for both the solution and its derivative to the initial value problem of the nonlinear VIDE.

This paper is organized in the following way. In Section 2, we introduce the Petrov-Galerkin finite element scheme for the problem (1.1) and recall some basic error estimates and asymptotic expansions obtained in our previous work. Section 3 is devoted to the study of the asymptotic expansions for the PGFE solutions and the iterated PGFE derivatives. In Section 4, we will investigate the Richardson extrapolation for the PGFE solutions and the iterated PGFE derivatives, and present some a-posteriori error estimators that use the superconvergent approximations obtained by post-processing the PGFE solutions with Richardson extrapolation. In Section 5 we will present some numerical examples to illustrate the theoretical results.

2. Petrov-Galerkin Finite Element Methods

In this section we will introduce the Petrov-Galerkin finite element methods and recall the global convergence results and asymptotic expansions for the L^2-projection operator obtained in [15] and [4], respectively. For this purpose, we first define a nonlinear integral operator $G : C(I) \to C(I)$ by

$$(G\varphi)(t) := f(t, \varphi(t)) + \int_0^t k(t, s, \varphi(s))ds.$$

Then, the problem (1.1) reads: Find $y = y(t)$ such that

$$y'(t) = (Gy)(t), \quad t \in I, \tag{2.1}$$

and its Petrov-Galerkin weak form consists in finding $y \in H_0^1(I)$ (and then $y' \in L^2(I)$) such that

$$(y', v) = (Gy, v), \quad \forall v \in L^2(I), \tag{2.2}$$

where $(\cdot, \cdot)$ denotes the usual inner product in $L^2(I)$ and $H_0^1(I) := \{v \in H^1(I) : v(0) = 0\}$ is the Sobolev space.

Let $T_h : 0 = t_0 < t_1 < \cdots < t_N = T$ be a given mesh for the interval I, and denote the finite element trial and test function spaces, respectively, by

$$S_m^{(0)}(T_h) := \{v \in H_0^1(I) : v|_{\sigma_k} \in P_m, 0 \le k \le N-1\}$$