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A NEW APPROACH TO SOLVE SYSTEMS OF LINEAR EQUATIONS*[1)]

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Abstract

We propose a new iterative approach to solve systems of linear equations. The new strategy integrates the algebraic basis of the problem with elements from classical mechanics and the finite difference method. The approach defines two families of convergent iterative methods. Each family is characterized by a linear differential equation, and every method is obtained from a suitable finite difference scheme to integrate the associated differential equation. The methods are general and depend on neither the matrix dimension nor the matrix structure. In this preliminary work, we present the basic features of the method with a simple application to a low dimensional system.

Key words: Iterative method, Linear systems, Classical dynamics.

1. Introduction

The new approach is based on the analysis of the motion of a damped harmonic oscillator in the gravitational field [1]. The associated equation of motion is

$$mX_{tt} + \alpha X_t + aX = b \tag{1}$$

where $X = X(t)$, is the one dimensional displacement of of a mass m under a dissipation ($\alpha > 0$), a harmonic potential ($a > 0$) and a constant acceleration (b, gravitational field). The total energy variation is given by the equation

$$\frac{dE}{dt} = -\alpha(\frac{dX}{dt})^2 \tag{2}$$

where

$$E = \frac{1}{2}m(\frac{dX}{dt})^2 + \frac{1}{2}aX^2 - bX \tag{3}$$

The solution of the motion equation (1) is given by the sum of two contributions: the homogeneous part and a particular solution:

$$X(t) = X_{hom.} + X_p \tag{4}$$

where

$$X_{hom.} = Me^{\lambda_1 t} + Ne^{\lambda_2 t} \tag{5}$$

being M and N constants.

$$X_p = \frac{b}{a} \tag{6}$$

that is

$$X(t) = Me^{\lambda_1 t} + Ne^{\lambda_2 t} + \frac{b}{a} \tag{7}$$

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The homogeneous component of the motion depends on the initial conditions $X(0), X_t(0)$, which are related to the two constants M, N, and the parameters λ_1 and λ_2 are either real negative or complex with a negative real part:

$$\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4am}}{2m} \tag{8}$$

For the critical case $\alpha^2 = 4am$, the two parameters λ are real and equal, and the solution is

$$X(t) = (M + Nt)e^{\frac{-\alpha t}{2m}} + \frac{b}{a} \tag{9}$$

In the three cases of possible values of λ we have that

$$lim_{t\to\infty} X_{hom.}(t) = 0 \tag{10}$$

Thus we get

$$lim_{t\to\infty} X(t) = \frac{b}{a} \tag{11}$$

This limit is the solution of the linear equation

$$aX = b \tag{12}$$

which, also, can be understood as the minimum of the potential $V(X) = \frac{1}{2}aX^2 - bX$, associated to the particle motion of mass m. In the above context, let us consider the following two remarks:

Remark. If $b = 0$, the solution of (12) is $X = 0$, which is the limit of the solution (7) and (9) when $t \to \infty$.

Remark. If $b = 0$ and $a = 0$, the limit of (7), when $t \to \infty$, is an arbitrary constant, depending on the initial conditions. This will be related to the undetermined systems.

Let us consider again the mechanical model (1) in the overdamped limit: the dominant effect is the dissipative one. Thus, by taking $\alpha = 1$ we get the equation:

$$X_t + aX = b \tag{13}$$

The equilibrium point of the dynamical system (13) is once more the solution of the linear equation (12). Such equilibrium point is stable.

The above mechanical considerations are the basis for the two new families of iterative methods to solve a system of linear equations. Each iterative method will be defined by a finite difference method to solve the linear equations (1) and (13). In the next two sections, we will extend the previous one dimensional mechanical considerations to the case of a system of equations.

An important feature of this approach is that we relate the problem of solving a system of linear equations to integrate the equation of motion of one particle, that tends asymptotically to a position which is identified with the solution of the above linear system. Thus, we can expect to have general iterative methods which need many iterations, but the convergence is satisfied while the discretization of the differential equation of motion satisfies conditions related to the conservation and variation of the energy of the system.

2. The Damped Methods

Let us consider the system of linear equations

$$AX = B \tag{14}$$

where A is a real $n \times n$ matrix, X and B real n-dimensional vectors. Also, we can interpret (14) as the extremum of the potential $V(X) = \frac{1}{2}X^T AX - X^T B$. We have three possibilities: one unique extremum, infinite extremums and no extremum, which correspond to one, infinite or no solutions for the system (14). In this context, we can consider the vectorial dynamical equation, similar to (1):

$$X_{tt} + \alpha X_t + AX = B \tag{15}$$

Key Assumption. Let A be a matrix with positive real spectrum. Then, we can consider a similarity transformation in the equation (15), and to reduce it to a system of n equations related to the type of (1), such that the positive eigenvalues of the matrix A guarantee that the asymptotic behaviour of the solution of (15) gives the solution of the system (14)