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SYMPLECTIC SCHEMES FOR QUASILINEAR WAVE EQUATIONS OF KLEIN-GORDON AND SINE-GORDON TYPE*

Xiao-wu Lu
(*Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA*)

Abstract

A class of finite difference methods of first- and second-order accuracy for the computation of solutions to the quasilinear wave equations is presented. These difference methods are constructed based on the symplectic schemes to the infinite-dimensional Hamiltonian system. Numerical experiments are presented to demonstrate the superior performance of these methods.

Key words: Symplectic integration, Sine-Gordon equations, Finite difference method.

1. Introduction

There has been much discussion recently about designing symplectic numerical schemes for both finite and infinite dimensional Hamiltonian systems ([1] - [8]). A class of symplectic schemes for linear wave equations has been suggested [1], and numerical experiments for these schemes have been conducted [8, 9]. Results show that the symplectic methods are inherently free from artificial dissipation and all kinds of non-Hamiltonian pollutions, and are thus of high quality and resolution.

In this paper, we try to generalize the discussion in the previous studies [1, 4, 5, 8], and construct symplectic schemes for one-dimensional quasilinear wave equation [11]

$$\partial_t^2 u - \partial_x f(\partial_x u) + g(u) = 0 \tag{1}$$

where g is a smooth function with $g(0) = 0$, $f^{'}(u) \geq \delta > 0$ and $uf^{''}(u) > 0$ for $u \neq 0$. Equation (1) models a vibrating string with an elastic external positional force, and has many applications in the study of nonlinear wave phenomena. Equations such as the Klein-Gordon, Sine-Gordon and the Φ^4 system may be all categorized within the system (1). The aim of this paper is to design numerical methods with high quality and resolution that can be used in the numerical computation of solutions for system (1). Since (1) can be written as an infinite-dimensional Hamiltonian system, it is our belief that the numerical methods that are constructed based upon the symplectic schemes will perform much better than other conventional methods, and thus will be good candidates for numerical simulation and calculation. The numerical methods we present here are constructed based on the similar procedures suggested in [1], and may be regarded as a generalized version of the schemes for the linear wave equation.

The paper is organized as follows: In section 2, we give a brief overview of general symplectic schemes and generating functional. In section 3, we construct a class of first- and second-order

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accurate difference methods for the quasilinear wave equations that result from the general schemes. The numerical results for these methods are presented in section 4, followed by conclusions in section 5.

2. Symplectic Schemes and Generating Functional

Symplectic schemes for both finite- and infinite-dimensional Hamiltonian systems have been widely discussed in the literature ([1]–[8]). In this section, we give a brief review of these schemes for infinite-dimensional Hamiltonian systems. Interested readers may consult the literature [1, 4, 6, 7] for details.

We consider $\mathcal{M}$ be a vector subspace of $C^\infty(R)\times C^\infty(R)$ consisting of those (u,v) with both $\mid u(x)\mid$ and $\mid v(x)\mid$ decreasing sufficiently rapidly such that the integral we write below is valid, and define $\mathcal{F}(\mathcal{M})=\{F(u,v)\mid F:\text{real-valued functional on }\mathcal{M}\}$. The symplectic structure on $\mathcal{M}$ is

$$\omega((u_1,v_1),(u_2,v_2))=\int_{-\infty}^{\infty}(u_1,v_1)J\,(u_2,v_2)^T\ dx. \tag{2}$$

where $J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. An infinite-dimensional Hamiltonian system on $\mathcal{M}$ takes the form

$$\partial_t u=-\frac{\delta H}{\delta v}(u,v,t),\qquad \partial_t v=\frac{\delta H}{\delta u}(u,v,t), \tag{3}$$

where $u=u(x,t), v=v(x,t)$ are two real functions, and $u(x,t),v(x,t)\in\mathcal{M}$ for any fixed t. $H(u,v)\in\mathcal{F}(\mathcal{M})$ is an energy functional in Hamiltonian mechanics. Within the Hamiltonian framework, the solution (u,v) of system (3) can be regarded as a Hamiltonian flow on the space $\mathcal{M}$ generated by a time-dependent map g^t, i.e., $(u(x),v(x))(t)=g^t\cdot(u_0,v_0)$, where $(u_0,v_0)=(u_0(x),v_0(x))\in\mathcal{M}$ is an initial condition. Furthermore, the map $g^t:\mathcal{M}\to\mathcal{M}$ is symplectic [10].

Since g^t is a symplectic map, it is naturally required that any numerical approximation to the system (3) should be carried out in such way that the discretized $(\Delta t,\Delta x)$-map should still be symplectic after the time and spatial discretization.

In this section, we will show how to construct a symplectic integrator (in the time direction) for system (3). First we will introduce a 4×4 Darboux matrix.

Definition [Darboux matrix]. *A 4×4 matrix T is called a Darboux if*

$$T^*\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}T=\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}, \tag{4}$$

where I is a 2×2 identity matrix, and $J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

In this paper, however, we will narrow down the class of Darboux matrices to a subclass of normal Darboux matrices, i.e., those which satisfy the conditions of $A+B=0$, $C+D=I$. Then the normal Darboux matrices can be characterized as

$$T=\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}J & -J\\ \frac{1}{2}(I+JL) & \frac{1}{2}(I-JL)\end{pmatrix} \tag{5}$$