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MULTIPLICATIVE SCHWARZ ALGORITHM WITH TIME STEPPING ALONG CHARACTERISTIC FOR CONVECTION DIFFUSION EQUATIONS*[1)]

Hong-xing Rui Dan-ping Yang
(*School of Mathematics and System Science, Shandong University, Jinan 250100, China*)

Abstract

Based on domain decomposition, we give two multiplicative schwarz methods with time stepping along characteristic for semi-linear, convection diffusion parabolic problems. we give some a priori error estimates, which tell us that the convergence of the approximate solution are independent of the iteration times at every time-level. Finally we give some numerical examples.

Key words: Multiplicative Schwarz method, Convection diffusion equation, Characteristic, Error estimate.

1. Introduction

Multiplicative Schwarz method, based on domain decomposition, is a powerful iteration methods for solving elliptic equations and other stationary problems. A systematic theory has been developed for elliptic finite element problems in the past few years, see [2, 5, 11, 12]. But there are little works of domain decomposition methods for time-dependence problems. In [11], Lions gives a kind of Schwarz alternating algorithm in two subdomain case for heat equations and gives a convergence result but does not give any error estimate. In [7, 8] Dawson and coworkers give a nonoverlapping domain decomposition method for parabolic equations, but since they use explicit schemes at intersection points, the stability condition $\Delta t \leq \frac{1}{2}H^2$ is needed. In [3, 4] Cai consider a kind of additive Schwarz algorithms and multiplicative Schwarz method and prove that the convergence rate is smaller than one for parabolic equations. In [13] the authors give the multiplicative Schwarz methods for linear parabolic problems.

In this paper we are interested in solving the convection diffusion problems using domain decomposition method. We use time-stepping along characteristic method mentioned by Douglas, Russell [9], which was powerful especially for convection-dominated equations, and Galerkin approximation in the space variables. At a fixed time level, the resulting equation is equivalent to an elliptic problem which depends on a time-step increment and the approximate solution at last time level. Therefore we can apply the multiplicative Schwarz method, originally proposed for elliptic equations to the convection diffusion equations at each time level. The crucial mathematical questions is then to know how the convergence and the error depend on the space mesh, the time step parameter and the number of iterations at each time level.

The outline of this paper is as follows. In Section 2 we present the continuous and discrete convection diffusion equations and give two kinds of multiplicative Schwarz algorithms with time-stepping along characteristic. We also give the optimal order error estimate results, which tell us that the approximate solutions converge after one cycle of iteration at each time level. In Section 3 we give some lemmas. In Section 4 we give proofs of theorems mentioned in Section 2. Finally in Section 5 we give some numerical examples.

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Throughout this paper, c or C ,with or without subscripts, denotes a generic, strictly positive constant. Its value may be different at different occurences, but is independent of the spacial meshsize h and time increment $\triangle t$, which will be introduced later.

2. Schwarz Algorithms and Convergence Results

Without loss of generality we consider the following model problem in a bounded polygonial domain $\Omega\subset R^2$

$$\frac{\partial u}{\partial t}+b\cdot\bigtriangledown u-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i})=f(u)\text{ in }\Omega, \tag{1}$$

$$u=0\text{ on }\partial\Omega, \tag{2}$$

$$u(x,0)=u^0(x)\text{ in }\Omega, \tag{3}$$

where $b=(b_1,b_2)$, $b\cdot\bigtriangledown u=b_1\frac{\partial u}{\partial x_1}+b_2\frac{\partial u}{\partial x_2}$, J=(0,T] denote the time interval, $a_{ij}=a_{ji}$ and there exists a positive constant γ such that

$$\sum_{ij=1}^{2}a_{ij}\xi_i\xi_j\geq\gamma|\xi|^2\quad \xi=(\xi_1,\xi_2)^\top\in R^2, \tag{4}$$

The standard variational formulation of the above problem is: Find $u(t)\in L^2(J;H_0^1(\Omega))$ such that

$$(\frac{\partial u}{\partial t},v)+a(u,v)=(f(u),v)\quad v\in H_0^1(\Omega), \tag{5}$$

$$(u(0),v)=(u^0,v),$$

where

$$a(u,v)=\int_\Omega\sum_{ij=1}^{2}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx, \tag{6}$$

$$(f(u),v)=\int_\Omega f(u)vdx.$$

Let $\triangle t$ denote time increment, $t^n=n\triangle t, u^n=u(t^n)$, for any point $x=(x_1,x_2)$, let $\bar{x}$ denote the point along the approximate characteristic direction,

$$\bar{x}=\begin{cases} x-b\triangle t=(x_1-b_1\triangle t,x_2-b_2\triangle t) & when\quad x-b\triangle t\in\Omega,\\ 2Y(x)-X(x) & when\quad x-b\triangle t\notin\Omega,\end{cases}$$

where $Y(x)\in\partial\Omega$ denote the project point of $x, X(x)\in\Omega$ denote the symmetric point of x about $\partial\Omega$. We also let

$$\overline{u}^{n-1}=u(\overline{x},t^{n-1}), \tag{7}$$

then[9,10,14]

$$\frac{u^n-\bar{u}^{n-1}}{\triangle t}=\frac{\partial u}{\partial t}+b.\bigtriangledown u+O(\frac{\partial^2 u}{\partial\tau^2}\tau), \tag{8}$$

where τ denote the unit vector at characteristic direction for the transfer term $(\frac{\partial u}{\partial t}+b.\bigtriangledown u)$. Then equation (1) can be approximated by

$$(\frac{u^n-\bar{u}^{n-1}}{\tau},v)=(f(u^{n-1}),v)+(\rho^n,v), \tag{9}$$