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TWO-SCALE FEM FOR ELLIPTIC MIXED BOUNDARY VALUE PROBLEMS WITH SMALL PERIODIC COEFFICIENTS*1)

Jin-ru Chen
(*School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing 210097, China*)

Jun-zhi Cui
(*ICMSEC, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, a dual approximate expression of the exact solution for mixed boundary value problems of second order elliptic PDE with small periodic coefficients is proposed. Meanwhile the error estimate of the dual approximate solution is discussed. Finally, a high-low order coupled two-scale finite element method is given, and its approximate error is analysed.

Key words: Two-scale FEM, Mixed boundary value, Small periodic coefficients.

1. Introduction

Composite materials have been widely used in high technology engineering as well as ordinary industrial products since they have many elegant qualities, such as high strength, high stiffness, high temperature resistance, corrosion resistance, and fatigue resistance. Most of the composite materials have small periodic configurations. Thus the static analysis of the structures of composite materials usually leads to the boundary value problems of elliptic partial differential equations with small periodic coefficients. Solving these problems by classical finite element methods is difficult because it usually requires very fine meshes and this lead to tremendous amount of computer memory and CPU time.

In order to solve this kind problem, many authors have proposed some useful methods, such as homogenization method, upscaling method, multiscale method, and so on (see [3] [5] [6] [7] [9] [10] [11] [12], and references therein). The two-scale method couples macroscopic scale and microscopic scale together, it not only reflects global mechanical and physical properties of structure, but also the effect of micro-configuration of composite material. Using this method, we can solve elliptic boundary value problems with small periodic coefficients by solving a homogenization problem with coarse meshes in whole domain and a periodic problem with fine meshes only in one small periodic subdomain.

However, most of authors only discussed pure Dirichlet boundary value problems. For mixed boundary value problems there are some difficulties. The main difficulty is that one can not obtain an asymptotic expression of the exact solution. It is well-known that mixed boundary value problems are very important in composite materials and applied mechanics. So we should study solving methods for the mixed boundary value problems arising from composite materials and the structures with small periodic configurations.

In this paper, we consider the mixed boundary value problem of second order elliptic equations with small periodic coefficients. Instead of asymptotic expression, we propose a dual

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approximate expression of the exact solution. The error of dual approximate solution is given. In order to couple macroscopic and microscopic analysis, we give a high-low order coupled two-scale finite element method, i.e., using high order finite elements with coarse grid size to approximate macroscopic problem in whole domain, while using low order finite elements with fine grid size to approximate microscopic problem in one small periodic subdomain. Based on this idea, the error of finite element approximate solution is obtained.

The remainder of this paper is outlined as follows: In §2 we introduce the problem and some notations. In §3 a dual approximate expression of the exact solution and its error are presented. In §4 a high-low order coupled finite element method is proposed and its error estimate is discussed.

In this paper, C (with or without subscripts) will denote a generic positive constant with possibly different values in different contexts. For any domain D, we use Sobolev space $W_p^m(D)$ with Sobolev norm $\|\cdot\|_{W_p^m(D)}$ and seminorm $|\cdot|_{W_p^m(D)}$ (see [1]). If $D=\Omega$, we omit D. Moreover if $D=\Omega$ and $p=2$, we denote the usual L^2 inner product by $(\cdot,\cdot)$, the Sobolev norm by $\|\cdot\|_m$ and seminorm by $|\cdot|_m$. Also we use Einstein summation notation, i.e., repeated index indicates to sum.

2. Preliminaries

Assume bounded domain $\Omega\subset\mathcal{R}^2$ to consist of entirely basic configurations, i.e., $\bar{\Omega}=\sum\limits_{z\in I^\epsilon}\epsilon(z+\bar{Q})$, where and hereafter ϵ is a small positive number,

$$
\begin{aligned}
I^\epsilon &= \{z\in\mathcal{Z}^2 \ | \ \epsilon(z+Q)\subset\Omega\},\\
Q &= \{y \ | \ 0<y_i<1, i=1,2\}.
\end{aligned}
$$

Consider the following problem

$$
\left\{\begin{array}{lll}
L^\epsilon u^\epsilon \equiv -\bigtriangledown\cdot(a^\epsilon\bigtriangledown u^\epsilon)=f, & \text{in} & \Omega,\\
u^\epsilon=0, & \text{on} & \Gamma_1,\\
a^\epsilon\bigtriangledown u^\epsilon\cdot n=g, & \text{on} & \Gamma_2,
\end{array}\right. \tag{2.1}
$$

where $a^\epsilon=(a_{ij}^\epsilon(x))$ is a bounded symmetric positive definite matrix of the functions with small period ϵ, f and g are sufficiently smooth functions, $n=(n_1,n_2)$ is a unit outer normal vector, $\Gamma_1=\partial\Omega\backslash\Gamma_2\neq\emptyset$, Γ_2 is defined by

$$
\Gamma_2=\bigcup_{z\in I_2^\epsilon}(\partial\epsilon(z+Q)\cap\partial\Omega), \quad I_2^\epsilon\subset I^\epsilon.
$$

Let $y=\frac{x}{\epsilon}$ and $a=(a_{ij}(y))=(a_{ij}^\epsilon(x))$, then a_{ij} is a periodic function with period 1. Assume $a_{ij}(y)$ to be one order smooth. First we introduce periodic function $N_k(y)$ which is the solution of the following equation

$$
\left\{\begin{array}{lll}
-\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial N_k}{\partial y_j})=\frac{\partial}{\partial y_i}a_{ik}, & \text{in} & Q,\\
N_k=0, & \text{on} & \partial Q.
\end{array}\right. \tag{2.2}
$$

Then we define a constant matrix $a^0=(a_{ij}^0)$ by

$$
a_{ij}^0=\int_Q(a_{ij}+a_{ik}\frac{\partial N_j}{\partial y_k})dy.
$$

Also we define function $u^0(x)$ to be the solution of the following equation

$$
\left\{\begin{array}{lll}
-\bigtriangledown\cdot(a^0\bigtriangledown u^0)=f, & \text{in} & \Omega,\\
u^0=0, & \text{on} & \Gamma_1,\\
a^0\bigtriangledown u^0\cdot n=g, & \text{on} & \Gamma_2.
\end{array}\right. \tag{2.3}
$$