Journal of Computational Mathematics, Vol.19, No.6, 2001, 583–590.

PICARD ITERATION FOR NONSMOOTH EQUATIONS[*1)]

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Abstract

This paper presents an analysis of the generalized Newton method, approximate Newton methods, and splitting methods for solving nonsmooth equations from Picard iteration viewpoint. It is proved that the radius of the weak Jacobian (RGJ) of Picard iteration function is equal to its least Lipschitz constant. Linear convergence or superlinear convergence results can be obtained provided that RGJ of the Picard iteration function at a solution point is less than one or equal to zero. As for applications, it is pointed out that the approximate Newton methods, the generalized Newton method for piecewise C^1 problems and splitting methods can be explained uniformly with the same viewpoint.

Key words: Nonsmooth equations, Picard iteration, Weak Jacobian, Convergence.

1. Introduction

Consider the following nonsmooth equations

$$F(x) = 0 \tag{1}$$

where $F : R^n \to R^n$ is Lipschitz continuous. A lot of work has been done and is beng done to deal with (1). It is basicly a generalization of the classic Newton method [8,10,11,14], Newton-like methods[1,18] and quasi-Newton methods [6,7]. As it is discussed in [7], the latter, quasi-Newton methods, seem to be limited when applied to nonsmooth case in that a bound of the deterioration of updating matrix can not be maintained without smoothness assumption of F at solution points. Therefore, more efforts have been made to discuss the former. The discussions are mainly on the local and global convergence under the assumption that F is semismooth.

An interesting discovery is that a majority of nonsmooth equations discussed in the recent years are almost either piecewise C^1 or well structured. The former mainly originate from nonlinear complementarity problems, variational inequality and nonlinear programming problems, see [4,12,15-17], while the latter from nonsmooth partial differential equations [5]. This encourages one to extend discussions on some specific problems which are either more than semismooth or well structured but non-semismooth.

In this paper, we are motivated to discuss these problems. We give a unified analysis on the generalized Newton method, approximate Newton methods and splitting methods from Picard iteration viewpoint.

In section 2, we simply review some basic definitions and results related to nonsmooth equations. A kind of generalized norm is introduced for the convex set-valued family. In section 3, we set up a relationship between the Lipschitz constant and the radius of the weak Jacobian of Picard iteration function. This is a generalization of the classic results in the smooth case. In

* Received December 9, 1997; Final revised January 15, 2001.
1)The research was partly supported by NNSFC(No. 19771047) and NSF of Jiangsu Province(BK97059).

section 4, we try to explain the approximate Newton methods, the generalized Newton method for piecewise C^1 problems and the splitting methods from Picard iteration viewpoint.

2. Generalized Jacobian

we use $\| \cdot \|$ to denote 2-norm of vectors in R^n and induced norm of matrices in $n \times n$ matrix space $L(R^n)$. We denote the set of points of R^n at which F is differentiable by D_F. We let $S(x,\delta)$ denote a closed ball in R^n with center x and radius δ.

We assume throughout that F is Lipschitz continuous in R^n in the sence that for every $x \in R^n$, there exist $L > 0$, and $\delta > 0$, such that

$$\| F(y) - F(z) \| \leq L \| y - z \| \tag{2}$$

for all $y, z \in S(x,\delta)$. Here L is called Lipschitz constant of F at x.

According to the Rademacher theorem, F is differentiable almost everywhere in R^n, and the generalized Jacobian of F at x was defined by Clarke [2] as follows:

$$\partial F(x) = conv[\lim_{x_i \to x, x_i \in D_F} \nabla F(x_i)].$$

Here and later, $\nabla F(x)$ denotes the Jacobian of F at $x \in D_F$, and "conv" denotes convex hull.

Proposition 2.1. *(Proposition 2.6.2, Clarke [2]) $\partial F(x)$ is compact and upper semicontinuous in the sence that for every $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in S(x,\delta)$,*

$$\partial F(y) \subset \partial F(x) + B,$$

where B denotes an open unit ball in $L(R^n)$.

A useful subset of $\partial F(x)$ was defined by Qi[13] as follows:

$$WF(x) = \lim_{x_i \to x, x_i \in D_F} \nabla F(x_i)$$

Here, we call it weak Jacobian. It is clear that

$$\partial F(x) = convWF(x)$$

Proposition 2.2. *(Proposition 2.1[18]) $WF(x)$ is compact and upper semicontinuous in the sense that for every $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in S(x,\delta)$,*

$$WF(y) \subset WF(x) + \varepsilon B,$$

where B denotes an open unit ball in $L(R^n)$.

Now consider convex sets in $L(R^n)$. Let A, B be two convex sets in $L(R^n)$, α is a scalar, define the operations

$$A + B = [c = a + b : a \in A, b \in B];$$

$$\alpha A = [c = \alpha a : a \in A].$$

Then, all these sets form a convex family with scalar multiplication and addition, we denote it by Γ. Taking an arbitrary element $A \in \Gamma$, we define

$$\| A \| = \sup_{\alpha \in A} \| \alpha \| .$$

We call $\| \partial F(x) \|$ the radius of the generalized Jacobian of F at x (RGJ in brief). Similarly, we can define the norm of $WF(x)$ by

$$\| WF(x) \| = \sup_{W \in WF(x)} \| W \| .$$

Proposition 2.3. *For every $x \in R^n$, $\| \partial F(x) \| = \| WF(x) \|$.*

Proof. It suffices to prove that

$$\| \partial F(x) \| \leq \| WF(x) \| .$$