Journal of Computational Mathematics, Vol.19, No.6, 2001, 629–638.

GENUINE-OPTIMAL CIRCULANT PRECONDITIONERS FOR WIENER-HOPF EQUATIONS[*1)]

Fu-rong Lin
(*Department of Mathematics, Shantou University, Shantou 515063, China*)

Abstract

In this paper, we construct the genuine-optimal circulant preconditioner for finite-section Wiener-Hopf equations. The genuine-optimal circulant preconditioner is defined as the minimizer of Hilbert-Schmidt norm over certain integral operators. We prove that the difference between the genuine-optimal circulant preconditioner and the original integral operator is the sum of a small norm operator and a finite rank operator. Thus, the preconditioned conjugate gradient (PCG) method, when applied to solve the preconditioned equations, converges superlinearly. Finally, we give an efficient algorithm for the solution of Wiener-Hopf equation discretized by high order quadrature rules.

Key words: Wiener-Hopf equations, Circulant preconditioner, Preconditioned conjugate gradient method, Quadrature rules, Hilbert-Schmidt norm.

1. Introduction

Wiener-Hopf equations are integral equations defined on the half-line:

$$\sigma x(t) + \int_0^\infty a(t-s)x(s)ds = g(t), \quad t \in \mathbb{R}^+,$$

where $\sigma > 0$, $a(\cdot) \in L_1(\mathbb{R})$ and $g(\cdot) \in L_2(\mathbb{R}^+)$. Here $\mathbb{R} \equiv (-\infty, \infty)$ and $\mathbb{R}^+ \equiv [0, \infty)$. In our discussions, we assume that $a(\cdot)$ is conjugate symmetric, i.e. $a(-t) = \overline{a(t)}$. Wiener-Hopf equations arise in a variety of practical applications in mathematics and engineering, for instance, in the linear prediction problems for stationary stochastic processes [8, pp.145–146], diffusion problems and scattering problems [8, pp.186–189]. In this paper, we consider using the preconditioned conjugate gradient (PCG) method to solve finite-section Wiener-Hopf equations:

$$\sigma x(t) + \int_0^\tau a(t-s)x(s)ds = g(t), \quad 0 \le t \le \tau. \tag{1}$$

Gohberg, Hanke and Koltracht [7] have introduced two circulant integral preconditioners, i.e., wrap-around and optimal circulant preconditioners to precondition the finite-section Wiener-Hopf equations (1). Circulant integral operators are operators of the form

$$(H_\tau y)(t) = \int_0^\tau h_\tau(t-s)y(s)ds, \quad 0 \le t \le \tau,$$

where $h_\tau \in L_1[-\tau, \tau]$ are τ-periodic, i.e., $h_\tau(t-\tau) = h_\tau(t)$ for $t \in [0, \tau]$. Let

$$A_\tau x(t) = \int_0^\tau a(t-s)x(s)ds, \quad 0 \le t \le \tau, \tag{2}$$

then the preconditioned equation is given by

$$(\sigma I + H_\tau)^{-1}(\sigma I + A_\tau)x_\tau(t) = (\sigma I + H_\tau)^{-1}g(t), \quad 0 \le t \le \tau. \tag{3}$$

* Received June 18, 1998; Final revised January 15, 2001.
1) Supported in part by the natural science foundation of China No. 19901017.

It has been proved in [7] that for sufficiently large τ, the spectra of the wrap-around and optimal circulant preconditioned operators are clustered around one. Hence the PCG method converges superlinearly for sufficiently large τ, see for instance [1].

In §2, we construct the genuine-optimal circulant integral preconditioner for the finite-section Wiener-Hopf equation (1). The genuine-optimal circulant preconditioner is the minimizer of the following Hilbert-Schmidt norm

$$|||I - (\sigma I + H_\tau)^{-1/2}(\sigma I + A_\tau)(\sigma I + H_\tau)^{-1/2}|||$$

over all circulant integral operator H_τ such that $(\sigma I + H_\tau)$ is positive definite. We prove that the genuine-optimal circulant preconditioners have the property that the spectra of the preconditioned operators $(\sigma I + H_\tau)^{-1}(\sigma I + A_\tau)$ are clustered around one for sufficiently large τ.

In this paper, we also consider the discretization of the preconditioned integral equations (3) by high order quadrature rules. Let the interval $[0, \tau]$ be partitioned into N equal subintervals of step-size $\iota = \tau/N$. By Newton-Cotes quadrature rules, using $s_k = k\iota,\ k = 0, 1, \cdots, N$ as quadrature points, we get the preconditioned matrix systems

$$(\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}(\sigma \mathbf{I}_p + \mathbf{A}_p\mathbf{D}_p)\mathbf{x}_p = (\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}\mathbf{g}_p, \tag{4}$$

where $p = N + 1$ is the number of quadrature points (if the rectangular rule is used, then the quadrature points are given by $s_k = k\iota,\ k = 0, 1, \cdots, N-1$ and $p = N$). Here $\mathbf{I}_p$ is the p-by-p identity matrix, $\mathbf{A}_p$ is the Hermitian Toeplitz matrix with the first column given by $(\iota a(0), \iota a(\iota), \cdots, \iota a(N\iota))^T$ and the $N \times N$ principal submatrix of $\mathbf{C}_p$ is a circulant matrix with the first column given by $(\iota h_\tau(0), \iota h_\tau(\iota), \cdots, \iota h_\tau((N-1)\iota))^T$. We recall that a matrix $\mathbf{A}_p$ is said to be a Toeplitz matrix if $\mathbf{A}_p = [a_{i,j}]$ satisfies $a_{i,j} = a_{i-j}$ and a matrix $\mathbf{C}_p$ is a circulant matrix if it is a Toeplitz matrix and $c_{-i} = c_{p-i}$ for $i = 1, 2, \cdots, p-1$. In (4), $\mathbf{D}_p$ is a diagonal matrix that depends only on the quadrature rule used. For instance, the diagonals of $\mathbf{D}_p$ corresponding to Simpson's rule are given by $(\frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \cdots, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3})$. We note that if $a(t)$ and $g(t)$ are smooth functions, then the accuracy of the discretized solutions $\sqrt{\sum_{i=0}^{N} \iota(x(i\iota) - [\mathbf{x}_p]_i)^2}$ of the rectangular, trapezoidal and Simpson's rule are of the order $O(\iota)$, $O(\iota^2)$ and $O(\iota^4)$ respectively, see [11] for instance.

We will give an efficient method to find an approximation of $(\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}$ in $O(p \log p)$ operations. We note that if the rectangular rule is used, (4) is basically a circulant preconditioned Toeplitz system which requires only $O(p \log p)$ operations in each iteration by means of FFTs [13] and the convergence rate of these systems has been analyzed for instances in [2, 9, 10, 14]. If high order quadrature rules are used, the discretization matrices of the circulant preconditioners $\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p$ are in general not circulant and it is difficult to find the inverse of $(\sigma \mathbf{I}_p + \mathbf{C}_p\mathbf{D}_p)^{-1}$ efficiently.

The outline of this paper is as follows. In §2, we construct the genuine-optimal circulant preconditioners for (1) and prove that the spectra of the preconditioned operators are clustered around one. Numerical results are given in §3 to illustrate the efficiency of circulant preconditioners. In §4, we propose an efficient algorithm for solving (4) and give numerical results to show the fast convergence and the stability of our algorithm.

2. Genuine-Optimal Circulant Integral Preconditioners

Let H_τ be a circulant preconditioner. We then solve the preconditioned equation

$$[(\sigma I + H_\tau)^{-1}(\sigma I + A_\tau)]x_\tau(t) = (\sigma I + H_\tau)^{-1}g(t).$$

A natural idea is to find the circulant operator H_τ such that the Hilbert-Schmidt norm

$$|||H_\tau - A_\tau|||^2 \equiv \int_0^\tau \int_0^\tau (a(t,s) - h_\tau(t-s))\overline{(a(t,s) - h_\tau(t-s))}dtds$$